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CALDER'S
ARITHMETIC
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FAMILIAR EXPLANATION

OF THE HIGHER PARTS OF

ARITHMETIC:

COMPRISING

FRACTIONS, DECIMALS, PRACTICE;

PROPORTION, AND ITS APPLICATIONS;

ETC. ETC.

WITH AN APPENDIX.

DESIGNED AS AN INTRODUCTION TO ALGEBRA.

BY THE

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PREFACE.

My object in printing this Work is to supply a want which I, as a Teacher, have experienced, of a School Book, which would explain the more difficult parts of Arithmetic in a manner sufficiently familiar for the comprehension of Youth, and yet sufficiently accurate and attentive to principles to be worthy of being considered an explanation.

It is well known to teachers, that pupils, in going through all the operations of Fractions, are generally taught to perform the mechanical process which the Rule tells them will bring out the desired *answer*; but not a particle of demonstration is given by which that process is shown to be correct. Consequently, boys make the grossest mistakes whenever the smallest knowledge of the principles of Fractions is required; and when any who have so learned arithmetic advance to algebra, they find all the operations required in the treatment of Fractions perfectly new to them: whereas I think that it will be found, that if young people of

both sexes are taught according to the rational method laid down in this book, such learners will not only obtain a portion of that mental training which is derived from mathematical study, but will, if they pursue that study, find the path thereto smoothed to a very considerable extent.

In the arrangement of the various subjects, I have, upon the whole, followed the plan adopted in THROWER'S EXAMPLES, a work published for the use of King Edward the Sixth's School, Birmingham, containing many thousands of examples, and which only needs to be known to be appreciated by every Schoolmaster.

According to this plan, Fractions, Decimals, and Practice are placed before Proportion and its applications; and there is no doubt that Proportion cannot be *understood* till Fractions are thoroughly mastered. But as a learner is, by this arrangement, kept back from Proportion too long for the wants of many scholars who leave school early, I have therefore explained the *principles* of Proportion under the head of Fractions, and have thereby left but little to be explained in Rule of Three, as it is commonly called, but the mere mechanical process of working Examples: so that those Teachers who wish to teach this Rule

before Fractions will find ready for them, under the head of Proportion, as much explanation as can be given to pupils unacquainted with Fractions. And afterwards, when the pupil goes through Fractions, he may comprehend the processes in Rule of Three, which he has hitherto performed in mere obedience to a RULE.

In teaching boys under the age of fourteen, I recommend the commencement of Fractions and Rule of Three at the same time, so that before the pupil comes to the difficult subject of Ratio and Proportion, he may have acquired a mechanical dexterity in working questions in Rule of Three; he will then the more readily receive the explanation of the subject which he will find in its proper place among Fractions.

In converting Circulating Decimals into equivalent Vulgar Fractions, I have preferred using the algebraical method, rather than, by working a few Examples, deduce a RULE for this conversion; for such a course would be only to teach pupils the common error of deducing the *general* rule from the *particular* Example. Also, the Algebraical method is very simple and intelligible to one who clearly understands the nature of Decimal notation. And in this volume I have, in all cases where either of two modes of operation might

be used, preferred the one which was the same as that required under similar circumstances in algebra; so that a pupil may not have to learn two methods of performing the same process.

In the Rule for Equation of Payments I have ventured to differ from the principle laid down by many standard writers on arithmetic, who contend that the common rule is not *strictly* correct, though sufficiently so for practical use. I have endeavoured to show that the view so taken is erroneous, and that the usual method is *perfectly* correct.

I hope that the Rule for the Extraction of the Cube Root will be found more simple than any that have hitherto been given. I have found in my own experience as a teacher, that pupils have remembered this method very readily, especially in algebraical examples.

The Appendix consists of a few detached articles which could not very conveniently have been introduced into the body of the work.

I have no doubt that some readers, who may be well acquainted with the subjects here treated of, will say that this book is too diffuse in its explanation; but I

think that this complaint will not come from those for whose use it is mainly designed, viz. youth, and teachers of youth: nor from those who, later in life, may think it worth their while to try and learn the meaning of those rules and operations, which they never comprehended while at school. If such persons can, without any assistance, understand what is here contained, I shall feel well satisfied with the success of my attempt.

Should this little book be well received, I shall publish another portion containing the Elementary Rules, upon the plan which has been here adopted, with this exception—that in addition to the explanation given of the elementary portions, I should always collect the substance of these explanations into a RULE; as I think it is convenient for younger pupils to have some prescribed form which they may commit to memory: and it will then be left to the discretion of the teacher, whether he will be content to use the mere RULE, or instil the accompanying explanation.

Since I have been anxious to keep this work within a moderate compass, I have confined it to the treatment of those branches which usually enter into School instruction; and have therefore omitted all mention of the nature and use of logarithms, or the method of

fixing the units of weight and measure, &c. A third portion may, perhaps, be prepared, suited for advanced pupils, and containing all that can be given upon these subjects without entering upon algebra, including also a short treatise upon Mensuration.

Each of the above portions would probably be equal to half the amount of the present work, and would be published separately; or the whole of the parts, if published, might be obtained in one volume.

In a work like this, which, as far as I know, is new in its plan, an author can hardly be expected to have completely succeeded: but I trust that no serious errors will be found. If there be such, I shall be glad to have them pointed out, and shall be happy to correct them, if a second edition should ever be called for.

F. C.

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THE FOLLOWING SIGNS MUST BE VERY WELL KNOWN BEFORE THIS BOOK IS READ.

$+$ *plus*, placed between two numbers, shows that they are to be added together.

$-$ *minus*, between two numbers, shows that the latter number is to be subtracted from the former.

$\times$ *into*, or *multiplied by*, between two numbers, shows that the two are to be multiplied together.

$\div$ *by*, or *divided by*, between two numbers, shows that the former is to be divided by the latter.

$=$ *equal to*, between two quantities, shows that the quantities on each side of it are equal.

$\therefore$ stands for *therefore*.

$\because$ *because*, or *since*.

$>$ *greater than*, placed between two quantities, shows that the former is greater than the latter.

$<$ *less than*, placed between two quantities, shows that the former is less than the latter.

Note.—The figures enclosed in parentheses, as (8) in p. 4, line 19, refer to previous articles in the work.

ARITHMETIC.

BEFORE commencing the consideration of Fractions, it will be necessary to explain certain properties of numbers, which enter very much into the treatment of fractions.

MEASURES AND MULTIPLES.

1. Of two numbers, the larger is said to be a *multiple* of the smaller, when the larger can be divided exactly, *i. e.* without remainder, by the smaller; and the smaller number is said to be a *measure* of the larger. Thus, since 12 can be divided exactly by 4, 12 is called a multiple of 4, and 4 a measure of 12. So, 16 is a multiple of 8, and 8 a measure of 16.

2. Also, one number is said to be a *common* multiple of two or more numbers, when it can be divided exactly by the two or more numbers. Thus, 24 is a common multiple of 6, 4, and 2, because it can be divided exactly by those numbers; and 12 is the *least* common multiple of 6, 4, and 2, because it is the *least* number that can be divided exactly by those three divisors.

3. A number is said to be a common measure of 2 or more numbers when it will exactly divide them both; and

the greatest number which will so divide them is c
their greatest common measure. Thus, 2, 4, 6, 8, 12
common measures of 24 and 36; and 12 is their gre
common measure.

We shall for the future write G. C. M. for greatest c
mon measure, and L. C. M. for least common multiple.

4. A number is called a *prime* number when it ca
be divided exactly by any number greater than 1. T
3, 5, 7, are prime numbers, because they have no div
greater than 1; we also call such numbers *primes.*

5. Two or more numbers are said to be prime to
another, when they cannot *all* be divided exactly by
number greater than 1. Thus 7, 8, 9, are prime to e
other: but such numbers are not necessarily themse
prime numbers; for 8 can be divided by 2 and 4; an
can be divided by 3; but 8 and 9 have no divisor comr
to both.

6. All numbers not prime are called *composite;* beca
they are composed of 2 or more prime numbers multip
together. Thus, 6 is a composite number, being forn
by the multiplication of the prime numbers 2 and 3.

7. The prime numbers which, when multiplied togetl
form a composite number, are called *factors* of t
number. Thus, 2, 2, 2, 3, are factors of 24, beca
$2 \times 2 \times 2 \times 3 = 24$. So the factors of 36 are 2, 2, 3
for $2 \times 2 \times 3 \times 3 = 36$.

8. To break up a composite number into its prime fact
divide it by the smallest prime which will divide it with
remainder; and continue dividing by prime divisors u
the last quotient is 1. The number will be found equal
the product of all the divisors—that is, all the divisors
its factors.

Ex. To break up 120 into prime factors:

$$\begin{array}{r|l} 2 & 120 \\ 2 & 60 \\ 2 & 30 \\ 3 & 15 \\ 5 & 5 \\ & 1 \end{array}$$

$\therefore$ 2, 2, 2, 3, 5, are its prime factors,
or, $2 \times 2 \times 2 \times 3 \times 5 = 120$.

When, therefore, I say—break any number, as 120, into its factors, I mean, exhibit it in the above form. This form is called an equation: thus, $7 \times 5 = 35$ is an equation; and 7×5 is called the left-hand *side*, and 35 the right-hand *side*, of the equation.

Obs. If any one number be divisible by a second, we must have all the factors of the second number among the factors of the first: thus, if 36 be divisible by 12, all the factors of 12 will be among the factors of 36, or 36 must contain all the factors of 12.

9. Also, if I wish to divide 35 by 5, (since 35 is the same as 7×5,) I may divide 7×5 by 5; and since I know that the quotient is 7, I can now observe that that quotient is found by taking away the divisor 5 out of the product 7×5: so also in the equation $7 \times 3 \times 5 = 105$, to divide the 105 by 5, take away the 5 out of the factors on the left-hand side, and there remains 7×3, or 21, as quotient; to divide by 7, take away 7, and 3×5, or 15, is quotient; or, we can divide by the product of 2 factors at once: thus, to divide by 15, take away 3×5, and the quotient is 7. This is a very quick method of dividing a composite number by any one or more of its factors, especially if the number be large.

Thus, $\because 2520 = 2 \times 2 \times 2 \times 7 \times 3 \times 3 \times 5$;
or $= 8 \times 7 \times 9 \times 5$;

therefore, if I wish to divide 2520 by any measure of it, as 24, I take away 24, or $2 \times 2 \times 2 \times 3$, from its factors, and the product of the remaining factors, $7 \times 3 \times 5 = 105$, gives the quotient. To divide 2520 by 63, I remove 7×9, and the quotient $= 8 \times 5 = 40$.

LEAST COMMON MULTIPLE.

10. It is required to find the L. C. M. of 2, 3, 5, 6, 9, 10, 12, 18, 20.

Now, of these numbers, 2, 3, 6, and 9 are divisors, or measures, of 18; every number, therefore, which is a multiple of 18, will also be a multiple of 2, 3, 6, 9. So likewise, every multiple of 20 will be a multiple of 5 and 10; $\therefore$ every multiple of 18 and of 20 will be a multiple of 2, 3, 6, 9, 5, and 10. If, then, I find the L. C. M. of 12, 18, and 20, that multiple will be the L. C. M. of 2, 3, 5, 6, 9, 10, 12, 18, 20. Now, $12 \times 18 \times 20$, or 4320 gives *a* multiple of 12, 18, and 20, since this product is plainly divisible by 12, 18, and 20; but this is not the *least* C. M.

11. Break the numbers 12, 18, 20, into their prime factors, as in (8), and place a comma between each set of factors, thus:

12,	18,	20,
$2 \times 2 \times 3$,	$2 \times 3 \times 3$,	$2 \times 2 \times 5$.

Now, by (8), in order that the L. C. M. may contain 12, it must contain all the factors of 12; $\therefore$ it must contain 2, 2, 3,

so, to contain 18, ... 2, 3, 3,

and to contain 20, ... 2, 2, 5;

i. e. in the required L. C. M. I want two 2's, two 3's, and one 5; but in the product of $12 \times 18 \times 20$, or of $2 \times 2 \times 3 \times 2 \times 3 \times 3 \times 2 \times 2 \times 5$, I have five 2's, and three 3's; $\therefore$ I have three 2's and one 3 which I do

not require; and, omitting these, the remaining factors, multiplied together, give $2 \times 2 \times 3 \times 3 \times 5 = 180$: where it is plain that this row of factors contains all that is necessary for 12, 18, and 20, and no more; and therefore the product of these factors is the least number divisible by 12, 18, and 20, or is their L. C. M.

12. The whole work should be written out as follows, where the mark ($-$) is placed over every number which is afterwards to be omitted:—

$$\bar{2},\ \bar{3},\ \bar{5},\ \bar{6},\ \bar{9},\ \bar{10},\ 12,\ 18,\ 20.$$

$$2 \times 2 \times \bar{3}, \qquad \bar{2} \times 3 \times 3, \qquad \bar{2} \times \bar{2} \times 5,$$

$$\text{L. C. M.} = 2 \times 2 \times 3 \times 3 \times 5 = 180.$$

Ex. II. To find the L. C. M. of 7, 16, 32, 21, 56, 42,

$$\bar{7},\ \overline{16},\ 32,\ \overline{21},\ 56,\ 42,$$

$$2 \times 2 \times 2 \times 2 \times 2, \qquad \bar{2} \times \bar{2} \times \bar{2} \times \bar{7}, \qquad \bar{2} \times 3 \times 7.$$

$$\text{L. C. M.} = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 7,$$

$$\text{or} = 32 \times 21 = 672.$$

In this example I first reject 7 and 21, because 42 contains them; next I reject 16, because 32 contains it; then, out of all the rest, I preserve five 2's, and reject the others: I preserve one 7, which is required for 56 and 42, and one 3, for 42. In preserving the five 2's, it is better to keep them all together, rather than have some in one set of factors, and some in another.

GREATEST COMMON MEASURE.

13. We have seen in (3) that the G. C. M. of two or more numbers is the largest divisor which can exactly be contained in those numbers.

Where the numbers are not large, this G. C. M. may be found by breaking them into their prime factors, and if

any factors are contained in all the numbers, the product of these will be the G. C. M.

Thus, to find G. C. M. of 36, 27, 144; breaking up into factors, we have—

36,	27,	144,
$2 \times 2 \times 3 \times 3$,	$3 \times 3 \times 3$,	$2 \times 2 \times 2 \times 2 \times 3 \times 3$.

Here it is plain that in these sets of factors two 3's, and no other number, are common to all; *i. e.* 3×3, or 9, is common to all the numbers 36, 27, 144,—and is therefore their G. C. M.

14. But where the numbers are large, and they cannot be easily broken into factors, the following rule is to be used:—

Take two of the proposed numbers, and divide the greater by the less: if there be a remainder, make that remainder a new divisor, and take the former divisor as a new dividend, and continue this process until there be no remainder: *the last divisor will be the* G. C. M.

If there be a third number, go through the same work with this third number, and the G. C. M. of the other two; then, as before, the last divisor will be the G. C. M. And the same process must be continued if there be 4, 5, &c., or any amount of numbers, of which we have to find the G. C. M.

We will try this rule upon the three numbers taken above, viz. 36, 27, and 144.

27) 36 (1	9) 144 (16
27	9
9) 27 (3	54
27	54

∴ 9 is G. C. M. of 27 and 36; and 9 is G. C. M. of 27, 36, and 144, as was shown above.

Ex. II. Find G. C. M. of 324, 456, 728.

```
324) 456 (1          12) 728 (60
     324                  720
     ---                  ---
 132) 324 (2           8) 12 (1
      264                  8
      ---                  -
   60) 132 (2           4) 8 (2
       120                  8
       ---                  -
  12)60 (5
     60
     --
```

∴ 12 is the G. C. M. of 324, 456; and 4 is the G. C. M. of 324, 456, 728.

15. If the last divisor be 1, we then learn that the numbers have no common divisor greater than 1; *i. e.* they are prime to each other.

Obs. Even when the divisors are less than 12, it is better to divide by long, rather than short division, because the remainders are thereby placed in the most convenient situation for continuing the process. This rule for finding the G. C. M. cannot be explained without the use of algebra.

16. When we have to find the L. C. M. or G. C. M. of any numbers which are not large, we may often, after a little experience, *see* the answer, without going through the work.

Thus, if it were required to find L. C. M. of 3, 4, 6, and 8, a pupil would soon learn that 24 was the required number. The easiest method of performing this operation, without writing, is to multiply in one's head the largest of the numbers given, by 2, 3, 4, and so on, until a number be found which will contain all the other numbers. So, in 3, 4, 6, and 8, if I try $2 \times 8 = 16$, it will not hold the 6; but $3 \times 8 = 24$ will hold the 6 as well as the 3 and 4, and therefore is the required L. C. M.

17. Again, to find G. C. M. of 4, 8, 12: it is easy to see that no number greater than 4 can go in 4, 8, and 12; and that 4 is their G. C. M.

When we can thus *see* the answer without working we are said to find the L. C. M. or G. C. M. by *inspection*.

OBS.—THE FOLLOWING ABBREVIATIONS WILL SOMETIMES BE USED.

Den^{r}	*for*	Denominator.
Num^{r}	„	Numerator.
Fr^{n}	„	Fraction.
Mult^{n}	„	Multiplication.
Add^{n}	„	Addition.
Sub^{n}	„	Subtraction.
Div^{n}	„	Division.
Quot^{t}	„	Quotient.
Rem^{r}	„	Remainder.
Diff^{e}	„	Difference.
Comp^{d}	„	Compound.
Red^{d}	„	Reduced.
Com.	„	Common.
Quan^{y}	„	Quantity.
Imp^{r}	„	Improper.
Frac^{l}	„	Fractional.
Ex.	„	Example.

FRACTIONS.

18. DEF. By the term *Unit*, we are to understand a single article of any kind, as 1 inch, 1 yard, 1 penny, 1 ounce, &c.

DEF. *Unity* is merely another name for the figure 1.

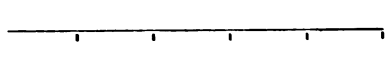
19. A Fraction is a part of a number, or quantity, supposed to be broken into any number of equal portions. If, then, the *unit* be divided into 4, 5, or 6 equal parts, one of these parts will be called one-fourth, one-fifth, or one-sixth, and is thus written—$\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$. So also, if any *two* of these parts be taken, the quantities thus taken will be called *two*-fourths, *two*-fifths, and *two*-sixths, and be written $\frac{2}{4}$, $\frac{2}{5}$, $\frac{2}{6}$.

20. The number below the line, which shows into how many parts the *unit* was broken, is called the *denominator*, because it expresses the denomination, or kind of parts, as *fourths, fifths, sixths.* The upper number, which *enumerates*, or counts, how many parts are taken, is called the *numerator*. Thus, in the fraction $\frac{2}{5}$, 5 is the denominator, 2 the numerator.

OBS. Such a fraction as I have been describing is called a *Vulgar Fraction.* We shall afterwards find that *Vulgar* fractions of a particular class can be expressed in another form, and are then called *Decimal* Fractions, or more commonly *Decimals.*

21. Again, any fraction, as $\frac{2}{5}$, may represent other quantities, besides being two-fifths of *one*, as we have just explained.

FIG. 1.



For if we divide the quantity 1, as 1 inch, into 5 equal parts (see Fig. 1), and also divide the quantity 2 into 5 equal parts, then one of the *latter* parts will be twice as great as one of the *former*; or *two* of the former and *one* of the latter have the same value. Now, the former or smaller is one-fifth of 1; and the latter or larger is one-fifth of 2; and, since two of the former = one of the latter, therefore two-fifths of 1 = one-fifth of 2, or $\frac{2}{5}$ of $1 = \frac{1}{5}$ of 2; and both these quantities are expressed by the fr^{n} $\frac{2}{5}$. Similarly, $\frac{6}{7}$ means *six*-sevenths of *one*, or *one*-seventh of *six*; $\frac{3}{4}$sh. means three-fourths of 1 shilling, or one-fourth of 3 shillings, each of which will, on trial, be found 9d.

22. From what has been said, it will be seen that the den^{r} of a fraction shows into how many parts the unit is divided; since, then, $\frac{6}{7}$ is proved to mean one-seventh of 6, *i. e.* it = six divided by 7; we hence see that a number placed as a den^{r}, underneath any other number as a num^{r}, shows that this den^{r} is to be taken as a divisor of the num^{r}; thus, $\frac{1}{7}{}^{5}$ implies that 15 is to be divided by 7: but as long as I do not work out the div^{n}, that division is said only to be *expressed.* Thus again, in the fraction $\frac{2}{7}{}^{3}$, we understand that 23 *is to be divided* by 7.

We shall hereafter see more clearly than now, that if 23 is divided by 7, the fraction $\frac{2}{7}{}^{3}$ may be called the *quotient.*

23. A fraction is called a *proper* fraction when the num^{r} is less than the den^{r}; thus $\frac{3}{8}$, $\frac{4}{5}$, are called *proper* fractions, because they really represent a part or parts of an unit, and are less than the whole unit.

24. A fraction whose num^{r} is equal to, or is greater than the denom^{r}, is called an *improper* fraction: Thus, $\frac{5}{5}$, $\frac{1}{7}{}^{2}$, are called *improper* fractions, because they are not in

reality *parts* of a unit broken up—*i. e.* are not less than the whole unit—but they are either one complete unit, or more than one.

25. Any whole number may be made to appear as an improper fraction by making 1 its denom^{r}: thus, $5 = \frac{5}{1}$, $10 = \frac{10}{1}$, &c.

26. A mixed number is one formed of a whole number and a fraction, as $3\frac{2}{5}$, which is read three and two-fifths—*i. e.* three units, and two-fifths of another unit; just as 3s. 7d. means 3 shillings and $\frac{7}{12}$ of another shilling, and might be written $3\frac{7}{12}$sh., where the unit is one shilling.

27. A fraction consisting of two or more fractions, connected by the word *of* placed between them, is called a *compound* fraction; as $\frac{2}{3}$ of $\frac{5}{7}$ of $\frac{8}{9}$: and a *complex* fraction is one in which either the num^{r} or den^{r}, or both, are fractions; as

$$\frac{2\frac{2}{4}}{4}, \quad \frac{3}{7\frac{1}{2}}, \quad \frac{1\frac{1}{3}}{3\frac{1}{5}}, \quad \frac{\frac{8}{9}\text{ of }1\frac{2}{3}}{7\frac{1}{5}}, \text{ &c.}$$

This last fraction is thus read: eight-ninths of one-and-two-thirds, divided by seven-and-one-fifth.

28. To multiply a fraction by any whole number, we multiply the num^{r} and retain the same den^{r}.

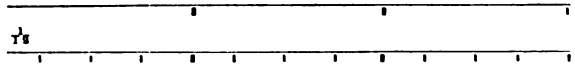
Thus, if it be required to multiply $\frac{5}{12}$ by 4,—just as 4 times 5 *pence* = 20 *pence*, so 4 times 5-*twelfths* = 20-*twelfths*, or $\frac{20}{12}$; $\therefore \frac{5}{12} \times 4 = \frac{20}{12}$, which $= \frac{5 \times 4}{12}$; *i. e.* to multiply a fraction by a whole number, the num^{r} is multiplied by the number, and the den^{r} is not altered.

29. Again, to multiply a fraction by any whole number, *if possible*, divide the den^{r} by this multiplier, and leave the num^{r} unaltered. By this method we should have

$$\frac{5}{12} \times 4 = \frac{5}{12 \div 4} = \frac{5}{3}.$$

To prove this let any unit be divided—1st, into 3 equal parts, and 2ndly, into 12 equal parts (see Fig. 2); then

Fig. 2.



each of the larger parts is called $\frac{1}{3}$, and each of the smaller $\frac{1}{12}$. Now, one of the *thirds* is four times as large as one of the *twelfths*, therefore 5 of such thirds are 4 times as large as 5 of the twelfths; *i. e.* $\frac{5}{3} = 4$ times $\frac{5}{12} = 4 \times \frac{5}{12}$; or, in the former shape, $\frac{5}{12} \times 4 = \frac{5}{3}$, which $= \frac{5}{12 \div 4}$; *i. e.* the den^{r} has been divided, and the num^{r} left unaltered.

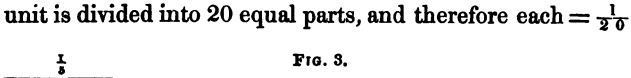
30. To divide a fraction by any whole number, divide the num^{r}, if possible, and keep the same den^{r}.

Thus, if it be required to divide $\frac{12}{13}$ by 4, just as 12 *pence* divided by 4 would give a quot^{t} 3 *pence*, so 12 *thirteenths* divided by 4, would give a quot^{t} 3 *thirteenths;* or $\frac{12}{13} \div 4 = \frac{3}{13}$ which $= \frac{12 \div 4}{13}$; *i. e.* the num^{r} is divided by the given divisor, and the den^{r} unaltered.

But, if we cannot divide the num^{r} of the fr^{n} by the whole number, then we must multiply the den^{r} by this divisor. By this method $\frac{3}{5} \div 4$ would $= \frac{3}{5 \times 4} = \frac{3}{20}$.

To prove this, observe in the fractions $\frac{3}{5}$ and $\frac{3}{20}$, that in the first fr^{n} the unit is divided into 5 equal parts, (see Fig. 3,) and therefore each $= \frac{1}{5}$; and, in the second, the same unit is divided into 20 equal parts, and therefore each $= \frac{1}{20}$:

Fig. 3.



now it is plain, from Fig. 3, that *one*-fifth, when divided by 4, becomes *one*-twentieth; so *three*-fifths divided by 4 become *three*-twentieths; *i. e.* $\frac{3}{5} \div 4 = \frac{3}{20}$, which $= \frac{3}{5 \times 4}$; hence we multiply the den^{r} by the given divisor, and leave the num^{r} unaltered.

Obs. Since it rarely happens that we can divide any num^{r} or den^{r} by any chance number, we therefore, in multiplying a fr^{n}, have, as a general rule, to multiply the num^{r}; and in dividing a fr^{n}, to multiply the den^{r}.

31. We have now shown that to multiply a fr^{n} by a whole number, we multiply the *num^{r}*, and to divide a fr^{n} we multiply the *den^{r}*; therefore if we multiply both num^{r} and den^{r} by the same quan^{y}, we shall have both multiplied and divided the fr^{n} by the same number; but since to multiply a quan^{y} by any number, and then to divide by the same, leaves the original quan^{y} unaltered, therefore to multiply both num^{r} and den^{r} of a fr^{n} will not alter its value:

$$\text{Thus, } \frac{3}{5} = \frac{3 \times 4}{5 \times 4} = \frac{12}{20};$$

and by observing Fig. 3, we shall see that since 3 of the large divisions, or *fifths*, $= 12$ of the small divisions, or *twentieths*, we have by inspection the same result, viz. $\frac{3}{5} = \frac{12}{20}$.

32. Since, then, any fr^{n}, as $\frac{6}{7} = \frac{6 \times 5}{7 \times 5} = \frac{30}{35}$, we may change the sides of this equation, and say $\frac{30}{35} = \frac{6}{7}$, which we know $= \frac{30 \div 5}{35 \div 5}$; *i. e.* the num^{r} and den^{r} of any fr^{n} may be divided by the same quan^{y}, and the fr^{n} will be unaltered in value.

CASE I.

TO REDUCE A FRACTION TO ITS LOWEST TERMS.

33. We have shown that both num^{r} and den^{r} of a fr^{n} can be divided by the same number without altering the value of the fr^{n}; and if the num^{r} and den^{r} cannot be exactly divided by any whole number, the fr^{n} is said to be in its lowest terms, because it is expressed in the lowest possible numbers.

Thus, $\frac{3}{4}$, $\frac{5}{6}$ are fr^{ns} in their lowest terms, because 3 and 4 in the first fr^{n}, and 5 and 6 in the second fr^{n}, have no com. divisor, or C. M.

But $\frac{8}{12}$ is not in its lowest terms, for both num^{r} and den^{r} can be divided by 4; and the fr^{n} will then be $\frac{2}{3}$. So also, in fr^{n} $\frac{1}{2}\frac{5}{5}$, 5 is G. C. M. of num^{r} and den^{r}; and the fr^{n} in its lowest terms $= \frac{3}{5}$.

It is plain, therefore, that if any fr^{n} be not in its lowest terms, and we wish so to reduce it, both num^{r} and den^{r} must be divided by the greatest possible divisor, *i. e.* by their G. C. M.

In working a sum under this case we should express the operation thus:

$$\frac{15}{25} = \frac{15 \div 5}{25 \div 5} = \frac{3}{5}$$

but when, in future cases, it is necessary to use this operation, it may be thus briefly written— $^{5)}\frac{1}{2}\frac{5}{5} = \frac{3}{5}$, where the small 5 is placed before the fr^{n} $\frac{1}{2}\frac{5}{5}$, to show what divisor has been used to reduce the fr^{n}.

34. When the num^{r} and den^{r} of a fr^{n} are both small numbers, the G. C. M. may be often found by *inspection*, or sometimes *some* com. measure may be found, though not

the greatest; and, by dividing by it, the fr^{n} may *partly* be reduced; and then a common measure of the fr^{n} so reduced may be found, and a second division performed, till we reduce the fr^{n} if not to the lowest, at any rate to lower terms.

Thus, $^{5)}\frac{3\,5}{2\,4\,5} = {}^{7)}\frac{7}{4\,9} = \frac{1}{7}$ is now reduced to lowest terms.

It is worth notice that every number ending in 5 or 0, must have 5 for a divisor, and every even number can be divided by 2.

But if mere inspection will not tell us the G. C. M. of num^{r} and den^{r}, it must be found by the method already given (14) for finding the G. C. M. of two given numbers.

CASE II.

TO REDUCE AN IMPROPER FRACTION TO A WHOLE OR MIXED NUMBER.

35. Ex. To reduce $\frac{17}{5}$ to a whole or mixed number. We have before shown (22) that this fr^{n} expresses that 17 is to be divided by 5: now, if the 17 were exactly divisible by 5, the quotient would be a whole number; but if not, as in this case, then since 15 is the largest multiple of 5 below 17, we can divide the 15 by 5, and the quotient is 3; but the division of the remaining 2 by the 5 cannot be *performed:* it must therefore be *understood,* by placing the divisor 5 underneath the 2 as a den^{r}, and thus $\frac{2}{5}$ may be said to be the quot^{t} of 2 when divided by 5: the whole quotient will, therefore, be 3 and $\frac{2}{5}$; and the operation may be thus written—

$$\frac{17}{5} = \frac{15+2}{5} = \frac{15}{5} + \frac{2}{5} = 3 + \frac{2}{5}.$$

or, as it is generally written, $= 3\frac{2}{5}$.

Also, since 5-fifths = 1, therefore 15-fifths = 3, and 17-fifths must = 3 and two-fifths, or $3\frac{2}{5}$, as just shown.

36. In writing the whole operation as fully as has been done above, it is necessary to find what is the rem^{r} after the num^{r} has been divided by the den^{r}. It must then be placed as the 2 has been placed in the above Ex., and the 15 is of course found by subtracting this rem^{r} from the whole num^{r}. Where the num^{r} and den^{r} are small, this rem^{r} may be found without working any div^{n} sum on the paper; but if the numbers are large, a div^{n} sum must be worked before the above operation can be shown.

Thus: Ex. 2. Reduce $\frac{327}{19}$ to a whole or mixed number.

19) 327 ($17\frac{4}{19}$
19
137
133
4
19

Here the rem^{r} being 4, the largest multiple of 19 below 327 is 323, which contains 19, 17 times,

therefore we have $\frac{327}{19} = \frac{323+4}{19} = \frac{323}{19} + \frac{4}{19} = 17\frac{4}{19}$;

or, briefly written, $\frac{327}{19} = 17\frac{4}{19}$.

CASE III.

TO REDUCE A MIXED NUMBER TO AN IMPROPER FRACTION.

37. Ex. To reduce $3\frac{2}{5}$ to an improper fr^{n}. That we may explain this operation properly, we must observe, that in order to express two or more numbers in one sum, these must all be of one kind or denomination; thus, in a simple addition sum, we add units to units, tens to tens, &c.; and in a compound addition sum we add together like quantities, as pounds to pounds, shillings to shillings, &c.

If, therefore, I have to express in one sum 3 units and 2 fifths, these two quantities must be reduced to the same kind. Now, I cannot reduce the fifths to a whole number, because two-fifths are less than unity; I must, therefore, reduce the 3 units to fifths, *i. e.* I must express 3 as a fr^{n} with a den^{r} 5.

$$\text{Thus, } 3 = \frac{3}{1} = \frac{3 \times 5}{1 \times 5} = \frac{15}{5}$$

adding then the 15-fifths to the two-fifths, I have the sum 17 fifths, or $\frac{17}{5}$.

The whole operation may be thus expressed:

$$3\tfrac{2}{5} = \frac{3}{1} + \frac{2}{5} = \frac{3 \times 5}{1 \times 5} + \frac{2}{5} = \frac{15}{5} + \frac{2}{5} = \frac{17}{5}.$$

38. It will be seen that this case is exactly the reverse of Case II.; and we may observe, that the num^{r} of the improper fr^{n}, viz. 17, is obtained by multiplying the whole number 3 by the den^{r} 5, and adding the former num^{r} 2: thus, in working the above Ex. I should say 3 times 5 are 15 and 2 are 17.

When, then, we have in future *cases* to reduce a mixed no., $3\frac{2}{5}$, to an improper fr^{n}, we should merely write $3\frac{2}{5} = \frac{17}{5}$, where all the intermediate work can be performed mentally, when the numbers are not large. We could thus write $9_{12}^{7} = \frac{115}{12}$, because 12 times 9 are 108 and 7 are 115.

CASE IV.

TO REDUCE FRACTIONS TO A COMMON DENOMINATOR.

39. It has been shown (37) that numbers, whether whole or fractional, cannot be added together without being reduced to the same denom^{n} or kind.

We must, therefore, show how to reduce fractions to a common den^{r}; and then they will be of the same denomination.

Ex. $\frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}$.

Now, in order that the fractions, when reduced, may still consist of as small numbers as possible, we shall always find their *least* common denom^{r}.

Also, since we can alter the form of fractions only by multiplying or dividing both num^{r} and den^{r}, we must therefore multiply these present den^{rs}, each one by some number, so that they all shall be changed into the new den^{r}; that is, the new den^{r} must be divisible by all these den^{rs}, and therefore must be their L. C. M.

Working according to (12) we find the L. C. M. of 4, 6, 8, 10, to be 120. We now choose such multipliers as shall make 4, 6, 8, 10, become 120: these will be 30, 20, 15, 12.

Now, we know (31) that if we multiply the den^{r} of a fr^{n} by any number, we must also multiply the num^{r} by the same number, or else the fr^{n} will be altered in value. We therefore work thus:

$$\frac{3}{4} = \frac{3 \times 30}{4 \times 30} = \frac{90}{120}$$

$$\frac{5}{6} = \frac{5 \times 20}{6 \times 20} = \frac{100}{120}$$

$$\frac{7}{8} = \frac{7 \times 15}{8 \times 15} = \frac{105}{120}$$

$$\frac{9}{10} = \frac{9 \times 12}{10 \times 12} = \frac{108}{120}$$

and these fractions may be thus written, $\frac{90, 100, 105, 108,}{120}$ showing at once that they have the same den^{r}.

40. In working sums of this kind, we should have to write down the process of finding the L. C. M. of all the den^{rs}, unless we could see it by *inspection*, which cannot be often done, when the numbers are large.

If it can be done, we must then merely write as the first line of the work L. C. M. = 120, or any number that it may chance to be.

CASE V.

TO ADD TOGETHER ANY NUMBER OF FRACTIONS.

41. These may be all proper fractions, or some of them may be proper, some improper, and some mixed numbers.

We have already seen that in order to add fractions together, we must reduce them to a com. den^{r}. Now, this has been done under Case IV.; and therefore we have only to conclude the operation in the last Ex. by performing the operation of adding the fractions so prepared. Now, we observe that whatever number of quantities of any kind are to be added together, the *kind* or denomination is not thereby changed: thus, 7 *months* added to 5 *months* would be 12 *months;* so, 4-*twelfths* + 3-*twelfths* = 7-*twelfths*, which, expressed in figures, gives $\frac{4}{12} + \frac{3}{12} = \frac{7}{12}$; hence it is plain that we have only to add the num^{rs} together, and retain the *same* den^{r}; therefore, so doing in the last Ex., we have

$$\text{sum of the fr}^{ns} = \frac{90+100+105+108}{120} = \frac{403}{120}$$

$$= 3\tfrac{43}{120} \text{ by (35)}.$$

If the proper fr^{n} $\frac{43}{120}$ had not been in its lowest terms, it must have been reduced to that state before the sum could be called complete.

42. If some of the fr^ns to be added together be mixed numbers, we must first add the frac^l parts precisely as above; and to this sum then add the amount of the whole numbers. Thus, if the Ex. which we have just worked had been as follows, $3\frac{3}{4}+\frac{5}{6}+6\frac{7}{8}+15\frac{9}{10}$ (where the insertion of the whole numbers 3, 6, and 15, is the only alteration), we should then have, as before,

$$\text{sum of the fractional parts} = 3\frac{43}{120};$$

therefore, adding in the whole numbers, we have the whole sum $= 3+6+15+3\frac{43}{120} = 27\frac{43}{120}$.

We will now work a second example completely.

Find the sum of $5\frac{2}{3}+1\frac{3}{4}+\frac{4}{7}+4\frac{5}{6}+\frac{7}{8}$.

$$\overline{3},\ \overline{4},\ 7,\ 6,\ 8,$$

$$7,\ \overline{2}\times 3,\ \ 2\times 2\times 2,$$

$$\text{L. C. M.} = 7\times 3\times 8 = 21\times 8 = 168.$$

$$\frac{2}{3}=\frac{2\times 56}{3\times 56}=\frac{112}{168}$$

$$\frac{3}{4}=\frac{3\times 42}{4\times 42}=\frac{126}{168}$$

$$\frac{4}{7}=\frac{4\times 24}{7\times 24}=\frac{96}{168}$$

$$\frac{5}{6}=\frac{5\times 28}{6\times 28}=\frac{140}{168}$$

$$\frac{7}{8}=\frac{7\times 21}{8\times 21}=\frac{147}{168}$$

$$\text{therefore, sum of frac}^{l}\text{ parts} = \frac{112+126+96+140+147}{168}$$

$$= \frac{621}{168} = 3\frac{117}{168} = 3\frac{39}{56} \text{ [in lowest terms,]}$$

$$\text{and whole sum} = 5+1+4+3\tfrac{39}{56} = 13\tfrac{39}{56}.$$

OBS. If there be any improper fractions in the proposed example, they should be reduced to mixed numbers before we begin to reduce to L. C. D.

CASE VI.

SUBTRACTION.

43. The examples in this Case will be of different kinds.

1st. When there are two fr^ns, both proper, as $\frac{3}{10} - \frac{1}{4}$, I.

2nd. When there are two fr^ns, one or both of which are mixed numbers, as $7\frac{3}{4} - 3\frac{1}{5}$, II.; $8\frac{2}{3} - 2\frac{5}{7}$, III.; $8 - 2\frac{5}{7}$, IV.

If there be any examples containing improper fr^ns, such examples may either be worked as I.; or, by reducing the improper fr^ns to mixed numbers, we bring them under II., III., or IV.

The difference between II., III., and IV. consists in this, that

in II. the former frac^l part $\frac{3}{4}$ is $>$ the latter $\frac{2}{7}$; (A)
in III. „ „ „ $\frac{2}{5}$ is $<$ „ $\frac{3}{7}$;
in IV. „ „ „ is 0, and $\therefore <$ $\frac{5}{7}$.

44. Under the head of Subtraction will also come examples in which there are more than two fractions; and wherein we have to find the difference between the sum of one set of fractions, and the sum of another set, as in the following

Ex. $5\frac{3}{5} + 4\frac{3}{8} - \frac{1}{2} + 3\frac{4}{8} - 7\frac{3}{7}$;

the signs + and − in this Ex. show that I am to take both $\frac{1}{2}$ and $7\frac{3}{7}$, that is, the sum of $\frac{1}{2}$ and $7\frac{3}{7}$, from the sum of $5\frac{3}{5} + 4\frac{3}{8} + 3\frac{4}{8}$.

DEF. If any numbers, either whole or frac^{l}, are joined together by one or more of the signs +, −, &c., the whole set of numbers thus connected is often termed an *expression.*

45. Just as it has been shown that fractions cannot be added together until they are reduced to a C. D., so it will appear that one fr^{n} cannot be subtracted from another till the two fr^{ns} are reduced to a C. D. Thus, if I had to take 3 pence from 1 shilling, I must bring the 1s. to pence, and then take 3d from 12d, obtaining a rem^{r} 9d.

Taking, then, Ex. I. $\frac{3}{10}-\frac{1}{4}$, we may see, by *inspection*, that L. C. D. $= 20$;

$$\text{therefore } \frac{3}{10}=\frac{3\times 2}{10\times 2}=\frac{6}{20};\quad \frac{1}{4}=\frac{1\times 5}{4\times 5}=\frac{5}{20};$$

$$\text{and}\quad \frac{3}{10}-\frac{1}{4}=\frac{6}{20}-\frac{5}{20}=\frac{6-5}{20}=\frac{1}{20}. \quad \text{(B)}$$

When the L. C. D. is not large, say under 150, a pupil will, with a little experience, soon be able at once to write down the line (B), without working the two previous operations on paper.

46. In Exs. II., III., IV. we must employ the same process that we use in Comp^{d} Sub^{n}. Take as patterns the three following:

A		B		C	
s.	*d.*	*s.*	*d.*	*s.*	*d.*
18	9	18	2	18	0
15	3	15	5	15	5
3	6	2	9	2	7

Now, in these Exs. it will be seen, that where the number of pence in the upper line exceeds that in the

lower line, as in (A), we can at once subtract the 3d. from the 9d., and the 15s. from the 18s., giving a complete rem^{r} 3s. 6d. So in Ex II., $7\frac{3}{4}-3\frac{2}{5}$, since $\frac{3}{4} > \frac{2}{5}$, therefore I can take the $\frac{2}{5}$ from the $\frac{3}{4}$, and the 3 from the 7.

The difference of the frac^{l} parts $=\frac{3}{4}-\frac{2}{5}=$(when reduced to L. C. D.) $\frac{15-8}{20}=\frac{7}{20}$; and diff^{e} of the whole numbers $=7-3=4$; therefore difference of complete fractions $=(7-3)+(\frac{3}{4}-\frac{2}{5})=4+\frac{7}{20}$,

$$=4\tfrac{7}{20}.$$

47. But in (B) and (C) the number of pence in the lower line is larger than in the upper, and in both cases I must borrow 1s. or 12d. from the 18s. in the upper line, and add this 12d. to the pence in the upper line, if there be any; and when I have sub^{d} the 5d. from the 14d. in (B), leaving rem^{r} 9d., and from 12d. in (C), leaving rem^{r} 7d., I subtract 15s. from 17s., (not 18s., because I have just borrowed 1s. from the 18s.) and in both cases I have a rem^{r} 2s.; therefore, the whole rem^{r} in (B) is 2s. 9d., and in (C) is 2s. 7d.

I now apply this process to Exs. III. and IV., viz. $8\frac{2}{5}-2\frac{3}{7}$, III.; and $8-2\frac{4}{7}$, IV. As, then, in the Comp^{d} Sub^{n} I borrowed 1s. from the higher den^{n} 18s., and added it to the pence in the upper row, if there were any, so I borrow 1 from the 8, and add it to the former fr^{n}, if there be any; thus in III. I add the 1 to the $\frac{2}{5}$, and then subtracting $\frac{3}{7}$, I say—

$$1\frac{2}{5}-\frac{3}{7}=\frac{7}{5}-\frac{3}{7}=\frac{49-15}{35}=\frac{34}{35},$$

then, taking the 8 as 7, I have $7-2=5$; therefore, showing the whole work at one view, I have

$$8\tfrac{2}{5}-2\tfrac{3}{7}=(7-2)+(1\tfrac{2}{5}-\tfrac{3}{7})=5+\tfrac{34}{35}=5\tfrac{34}{35}.$$

I place the parentheses in this last line to show that the $(7-2)$ gives 5, and that $(1\frac{2}{5}-\frac{3}{7})$ gives $\frac{3}{3}\frac{4}{5}$; but they need not be used in working Exs.

48. In Ex. IV., $8-2\frac{5}{7}$, I borrow 1 from the 8, but as I have no former fr^{n} to which I may add this 1, as in Ex. III., therefore subtracting the 2nd fr^{n} $\frac{5}{7}$ from the 1, I have

$$1-\frac{5}{7}=(\text{when red}^{d}\text{ to L. C. D.})\ \frac{7}{7}-\frac{5}{7}=\frac{7-5}{7}=\frac{2}{7};\quad (C)$$

and taking 7 from 8, as before, I have $7-2=5$; therefore, writing the whole as one process

$$8-2\tfrac{5}{7}=(7-2)+(1-\tfrac{5}{7})=5+\tfrac{2}{7}=5\tfrac{2}{7}.$$

49. In the line (C) I may observe, that when the number 1 is expressed as a fr^{n}, its num^{r} is made the same as the den^{r} of the fr^{n} whlch I have to subtract: thus, 7 is the new num^{r}, and is also the den^{r} of the original fr^{n}; whether, therefore, I subtract the smaller num^{r} from the new num^{r} or from the smaller fr^{n} of the den^{r}, I obtain the same result, viz. a rem^{r} 2; and I might say at once $1-\frac{5}{7}=\frac{2}{7}$, where the 2 has been found by subtracting the num^{r} 5 from its own den^{r} 7. Working thus, I should have written the whole Ex. as follows:

$$8-2\tfrac{5}{7}=(7-2)+(1-\tfrac{5}{7})=5\tfrac{2}{7},$$

only that in common use I should not write the two parentheses.

50. It was said in line (A) $\frac{3}{4}>\frac{3}{5}$; a beginner cannot see this at once; but he may do so very readily—thus: we already know that no two quantities can be compared, to see which is the larger, unless they be of the same kind; so, two fr^{ns} cannot be compared unless reduced to a C. D.

Any C. D. will, however, do for the purpose of merely telling which of the two fr^ns is the larger; now the product of the two den^rs is always *one* C. D., but not always the *least;* therefore, since 35 is a C. D. of $\frac{2}{7}$ and $\frac{3}{5}$,

$$\frac{2}{7} - \frac{3}{5} = \frac{2 \times 5 - 7 \times 3}{7 \times 5} = \frac{10 - 21}{35}; \quad \text{(C)}$$

and since $21 > 10$, therefore $\frac{21}{35} > \frac{10}{35}$; and $\frac{3}{5} > \frac{2}{7}$.

Now this result may be very rapidly obtained without any work on paper; for if we observe line (C), we notice that the numbers 10 and 21, which numbers alone I wish to compare, are found by multiplying the num^r of the 1st fr^n by the den^r of the 2nd, and the den^r of the 1st by the num^r of the 2nd. Thus, beginning with the num^r of the 1st, I say, *mentally*, $2 \times 5 = 10$: and again, $3 \times 7 = 21$; therefore, since the second product is greater than the first, so also the second fr^n is greater than the first.

Taking one or two more Exs. of this process—

1. To compare $\frac{3}{7}$ and $\frac{2}{9}$: I say, $3 \times 9 = 27$; $7 \times 2 = 14$; therefore the former fr^n is the larger.

2. To compare $\frac{3}{8}$ and $\frac{7}{12}$: I say, $3 \times 12 = 36$; $8 \times 7 = 56$; therefore the latter fr^n is the larger.

It will be noticed that the C. D. to which we have just now mentally reduced $\frac{3}{8}$ and $\frac{7}{12}$ is 96; but when we come to the *actual sub^n*, we shall not use *this* C. D., but the L. C. D. 24.

51. Lastly: when there are more than 2 fractions, as in

Ex. V. $5\frac{2}{3} + 4\frac{3}{8} - \frac{1}{2} + 3\frac{1}{6} - 7\frac{2}{7}$.

Here, omitting for the present the whole numbers, I have to subtract $\frac{1}{2}$ and $\frac{3}{7}$ from the sum of $\frac{2}{9}+\frac{3}{8}+\frac{5}{6}$: since the two fr^ns^ to be subtracted are together less than the three from which I have to subtract them, no difficulty will occur, and I proceed as in an Ex. in Addition, taking care to arrange the fr^ns^ to be subtracted last:

$$\frac{2}{9}+\frac{3}{8}+\frac{5}{6}-\frac{1}{2}-\frac{3}{7};$$

$$9,\ 8,\ 6,\ \bar{2},\ 7.$$

$$3\times 3,\quad 2\times 2\times 2,\quad \bar{2}\times\bar{3},\quad 7.$$

$$\text{L. C. D.} = 9\times 8\times 7 = 72\times 7 = 504.$$

$$\frac{2}{9} = \frac{2\times 56}{9\times 56} = \frac{112}{504} \qquad \text{See Art. (9).}$$

$$\frac{3}{8} = \frac{3\times 63}{8\times 63} = \frac{189}{504}$$

$$\frac{5}{6} = \frac{5\times 84}{6\times 84} = \frac{420}{504}$$

$$\frac{1}{2} = \frac{1\times 252}{2\times 252} = \frac{252}{504}$$

$$\frac{3}{7} = \frac{3\times 72}{7\times 72} = \frac{216}{504}$$

$$\therefore\ \frac{2}{9}+\frac{3}{8}+\frac{5}{6}-\frac{1}{2}-\frac{3}{7} = \frac{112+189+420-252-216}{504}$$

$$= \frac{721-468}{504} = \frac{253}{504}$$

therefore, bringing in the whole numbers with their proper signs,

$$\text{the whole expression} = 5+4+3-7+\frac{253}{504} = 12-7+\frac{253}{504}$$

$$= 5\tfrac{253}{504}.$$

52. We will take one more example of this kind.

Ex. VI. $3\frac{4}{5}-4\frac{1}{2}+7\frac{2}{7}-\frac{3}{4}$.

Now here $\frac{1}{2}+\frac{3}{4}$ are $> \frac{4}{5}+\frac{2}{7}$; but since this cannot be seen by mere *inspection*, I proceed as in Ex. V., as though I did

not know this, and provide for the difficulty at the proper time. Placing the fr^ns^ to be subtracted last, I have

$$\frac{4}{5}+\frac{2}{7}-\frac{1}{2}-\frac{3}{4}$$

5, 7, 2, 4,

L. C. D. $= 5\times7\times4 = 140.$

$$\frac{4}{5}=\frac{4\times28}{5\times28}=\frac{112}{140}$$

$$\frac{2}{7}=\frac{2\times20}{7\times20}=\frac{40}{140}$$

$$\frac{1}{2}=\frac{1\times70}{2\times70}=\frac{70}{140}$$

$$\frac{3}{4}=\frac{3\times35}{4\times35}=\frac{105}{140}$$

therefore $\frac{4}{5}+\frac{2}{7}-\frac{1}{2}-\frac{3}{4}=\frac{112+40-70-105}{140}$

$$=\frac{152-175}{140} \qquad \text{(D)}$$

Since, then, $175 > 152$, it now appears that I ought to have borrowed 1; but rather than begin the work again, I proceed thus: from the $\frac{152}{140}$ in line (D) I take as much of the $\frac{175}{140}$ as I can, viz. $\frac{152}{140}$, and there will remain $\frac{23}{140}$, which cannot be subtracted; but if I now borrow 1, I can subtract this $\frac{23}{140}$ from the 1, and remember to count the first (+) whole number in Ex. VI. (*i. e.* the 3) as one less than it now stands. The whole operation may be thus written, beginning from the line before (D):

$$\frac{4}{5}+\frac{2}{7}-\frac{1}{2}-\frac{3}{4}=\frac{112+40-70-105}{140}$$

$$=\frac{152-175}{140}\left(=\frac{152-152-23}{140}\right)$$

$$=\left(\frac{0-23}{140}\right)=\frac{-23}{140}$$

therefore borrowing the 1, $\quad 1-\frac{23}{140}=\frac{140}{140}-\frac{23}{140}=\frac{117}{140}$

and writing 2 for 3 in Ex. VI., and inserting the whole numbers with their proper signs,

the whole expression $= 2 + 7 - 4 + \frac{117}{140} = 9 - 4 + \frac{117}{140}$
$= 5\frac{1}{1}\frac{1}{4}\frac{7}{0}$.

The fractions inclosed in parentheses may in practice be omitted.

CASES VII.—IX.

MULTIPLICATION.

53. We shall show, as we proceed, that the above Cases are identical, *i. e.* that the word *of* and the sign × placed between fr^ns^, have the same meaning: thus, I have to prove that

$\frac{5}{7}$ of $\frac{2}{3}$ of $3\frac{3}{5}$ and $\frac{5}{7} \times \frac{2}{3} \times 3\frac{3}{5}$ have the same value. (E)

The pupil must here clearly call to mind the method of multiplying and dividing fractions by whole numbers.

I will first begin with a pair of fractions, as $\frac{5}{7} \times \frac{2}{3}$.

It has been proved that $\frac{2}{3}$ means *one-third* of 2; therefore I have now to multiply $\frac{5}{7}$ by one-third of 2: hence the product ought to be one-third of the product found by multiplying by 2 alone: *i. e.* if I multiply by 2, and then take one-third of that product, I shall obtain the correct result.

Multiplying $\frac{5}{7}$ by 2, I have $\frac{5}{7} \times 2 = \frac{5 \times 2}{7}$;

and since to take one-third of any quantity is to divide it by 3, I divide this last fr^n^ by 3, and have

$$\frac{5}{7} \times \frac{2}{3} = \frac{5 \times 2}{7} \div 3 = \frac{5 \times 2}{7 \times 3}. \quad \text{(F)}$$

Commencing now with the expression $\frac{5}{7}$ of $\frac{2}{3}$: we know that $\frac{5}{7}$ means 5 times one-seventh; therefore, if I have to take five-sevenths of any quantity, I first take *one*-seventh of it, and then multiply the quot^{t} obtained by 5: *i. e.* to take $\frac{5}{7}$ of $\frac{2}{3}$, I must divide the $\frac{2}{3}$ by 7, and multiply it by 5: hence, according to our rules of mult^{n} and div^{n} of fractions by whole numbers, I multiply the den^{r} of the $\frac{2}{3}$ by 7, and the num^{r} by 5, and have

$$\frac{5}{7} \text{ of } \frac{2}{3} = \frac{5 \times 2}{7 \times 3}. \quad \text{(G)}$$

Comparing (F) and (G), I now perceive that

$$\frac{5}{7} \text{ of } \frac{2}{3} = \frac{5}{7} \times \frac{2}{3}.$$

54. The following explanation may, perhaps, be intelligible to some pupils :

In multiplying a quantity, whether whole or fractional, by a whole number, I repeat that quantity as many times as there are units in the multiplier:

Thus, 7×5, or 7 times 5, means that I am to repeat the 5 as many times as there are units in 7;

So, $7 \times \frac{5}{8}$ shows that I am to repeat $\frac{5}{8}$, as often as there are units in 7;

Also, $1 \times \frac{5}{8}$ shows that I am to repeat $\frac{5}{8}$ as often as there are units in 1, *i. e.* 1 time, or once:

Therefore, $\frac{3}{4} \times \frac{5}{8}$ ought to mean, that I am to repeat $\frac{3}{4}$ as often as there are units in $\frac{5}{8}$.

Now, $\frac{3}{4}$ does not contain 1 unit,—it only contains $\frac{3}{4}$ of 1 unit; therefore, $\frac{3}{4} \times \frac{5}{8}$ is $\frac{5}{8}$ repeated $\frac{3}{4}$ of once, *i. e.* it $=$ $\frac{3}{4}$ of once $\frac{5}{8}$, or $\frac{3}{4}$ of $\frac{5}{8}$; and this is the required result, as in last article.

55. From what has been shown in (53), we may see that

since $\quad \frac{5}{7} \times \frac{2}{3} = \frac{5 \times 2}{7 \times 3} = \frac{10}{21}$,

and $\quad \frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$;

therefore $\quad \frac{5}{7} \times \frac{2}{3} = \frac{2}{3} \times \frac{5}{7}$:

and since the sign ($\times$) and the word *of* between frⁿˢ have been shown to have the same meaning, we see that

$$\frac{5}{7} \text{ of } \frac{2}{3} = \frac{2}{3} \text{ of } \frac{5}{7}.$$

56. Proceeding with the expression (E) in (53), and changing $3\frac{3}{5}$ into an impʳ frⁿ, we have

$$\frac{5}{7} \text{ of } \frac{2}{3} \text{ of } 3\tfrac{3}{5} = \frac{5 \times 2}{7 \times 3} \text{ of } \frac{18}{5};$$

and since $\frac{5 \times 2}{7 \times 3}$ is one frⁿ, and $\frac{18}{5}$ another frⁿ, I may use the method of multiplying, which has been shown to be true, for *two* fractions;

$$\text{therefore } \frac{5}{7} \text{ of } \frac{2}{3} \text{ of } 3\tfrac{3}{5} = \frac{5 \times 2 \times 18}{7 \times 3 \times 5}: \text{ (H)}$$

and if the expression in (E) had consisted of four frⁿˢ connected by *of* or ($\times$), the first three might have been reduced to one frⁿ, as in (G), and we should then have had but *two* frⁿˢ, which we have just shown how to reduce to one. Therefore the results proved in lines (E) (F) (G) are true, whatever number of frⁿˢ may be contained in any compound fraction.

57. To return to (H). It has been proved that if a numʳ and denʳ have any common measure, both of them may be divided by this common measure; therefore if in the right-hand side of equation (H), we find any factors common to both numʳ and denʳ, these should be struck out before we complete the multiplication.

We see in this case that there is a factor 5 in the numʳ and denʳ; also, that 18 in the numʳ and 3 in the denʳ have a common divisor 3: dividing by these common factors,

the expression stands thus $\dfrac{2 \times \overset{1}{\cancel{5}} + \overset{6}{\cancel{18}}}{\underset{1}{\cancel{3}} \times 7 \times \underset{1}{\cancel{5}}}$.

The remaining factors in the num^{r} are 2, 1, 6; and in den^{r} are 1, 7, 1; these form a fraction $\frac{2 \times 6}{7} = \frac{12}{7} = 1\frac{5}{7}$.

This process of striking out factors from num^{r} and den^{r} is called *cancelling*, and is to be continued as long as any factors can be found common to both num^{r} and den^{r}. When the numbers are large, this cancelling cannot be readily performed except by those who have had some experience.

Obs. A factor 1 left in a num^{r} or den^{r} will not alter the value of the other factors which remain, when we multiply all those in the num^{r}, and all those in the den^{r}, and hence the figures 1 need not be written, as in the above Ex.; but it must be noticed that if no other factors remain, the value of the num^{r} or den^{r} will consist of the product of as many figures 1 as there were factors in the num^{r} or den^{r} before cancelling; *i. e.* it will be = 1. If, therefore, all the factors in any num^{r} or den^{r} have been cancelled, the num^{r} or den^{r} of the fr^{n} after cancelling will be 1.

Ex. II. $8\frac{3}{5} \times 35 \times \frac{17}{24} = \frac{\overset{7}{\cancel{42}}}{\cancel{5}} \times \frac{\overset{7}{\cancel{35}}}{1} \times \frac{17}{\underset{4}{\cancel{24}}}$

$$= \frac{7 \times 7 \times 17}{4} = \frac{833}{4} \quad (1)$$

$$= 208\tfrac{1}{4}.$$

Here 42 and 24 had a c. m. 6, and 5 and 35 had a c. m. 5.

The fr^{n} $\frac{833}{4}$ in line (1) will be in its lowest terms, if all the factors common to num^{r} and den^{r} were cancelled at the proper time.

Ex. III. $\frac{7}{8}$ of $7\frac{1}{2}$ of $\frac{4}{15}$ of $\frac{2}{7} = \frac{\cancel{7}}{\underset{2}{\cancel{8}}} \times \frac{\cancel{15}}{\cancel{2}} \times \frac{\cancel{4}}{\cancel{15}} \times \frac{\cancel{2}}{\cancel{7}}$.

Here there are no factors to be *seen* in the num^{r}; but we must remember that in reality there have been 4 divisions of the num^{r} by the factors 7, 15, 4, and 2; and therefore 4 quot^{ts}, which in this case are each $= 1$; therefore the real num$^{r} = 1 \times 1 \times 1 \times 1$, or 1; so the quot^{ts} in the den^{r} are 2, 1, 1, 1, and their product $= 2$; therefore the reduced fraction 2 has 1 as num^{r} and 2 as den^{r}, *i. e.* it is $\frac{1}{2}$.

CASES VIII.—X.

DIVISION.

58. Case X. is the same as Case VIII. in principle—it merely uses a different method of expressing the process of Division, and includes some examples requiring both Multiplication and Division.

$$\text{Thus: } \frac{11}{12} \div \frac{5}{9} \text{ and } \frac{\frac{11}{12}}{\frac{5}{9}}$$

have the same meaning, both being read $\frac{11}{12}$ divided by $\frac{5}{9}$.

We have now to inquire how to perform this division.

We know by (31) that $5 = 9 \times \frac{5}{9} = 9$ times $\frac{5}{9}$, or that 5 is 9 times as large as $\frac{5}{9}$. If, now, I divide $\frac{11}{12}$ by 5 (when my object was to divide by $\frac{5}{9}$,) I shall divide by a quantity 9 times too large, and therefore my quotient will be 9 times too small; to obtain, then, the *true* quotient, I must multiply the first quot^{t} by 9; *i. e.* the correct quot^{t} is found by performing two operations—first, by dividing by 5, and secondly, by multiplying by 9; and performing these two operations at one step, we have

$$\frac{11}{12} \div \frac{5}{9} = \frac{11}{12} \times \frac{9}{5} \qquad \text{(K)}$$

and by comparing the two sides of this equation, we see that the divisor $\frac{5}{9}$ has been inverted into $\frac{9}{5}$, and the $(\div)$ has been changed into $(\times)$, or the div^{n} changed into multiplication.

If mixed or whole numbers are found in any expression which is to be simplified, we must, of course, reduce to imp^{r} fr^{ns}, and invert the divisor, as before. Thus

$$\frac{9}{14\frac{5}{8}} = \frac{\frac{9}{1}}{\frac{117}{8}} = \frac{9}{1} \times \frac{8}{117} = \frac{9}{1} \times \frac{8}{\cancel{117}^{13}} = \frac{8}{13}.$$

59. The following is a more complicated Ex. involving both Mult^{n} and Division.

Reduce to the simplest form $\frac{7\frac{3}{5}}{8\frac{5}{9}}$ of $\frac{6\frac{6}{11}}{8\frac{5}{9}}$.

Here we reduce mixed numbers to imp. fr^{ns}—change $\div$ into $\times$, and invert the divisors, as was shown in last art.; and then, making two steps in the operation,

$$\text{the expression} = \frac{\frac{38}{5}}{\frac{77}{9}} \times \frac{\frac{77}{12}}{\frac{53}{6}} = \frac{38}{5} \times \frac{9}{77} \times \frac{77}{12} \times \frac{6}{53} \quad \text{(L)}$$

$$= \frac{\cancel{38}^{19}}{5} \times \frac{9}{\cancel{77}^{11}} \times \frac{\cancel{77}^{11}}{\cancel{12}^{2}} \times \frac{\cancel{6}^{1}}{53} \quad \text{(M)}$$

$$= \frac{19 \times 9}{5 \times 53} = \frac{171}{265}.$$

In working such Exs. I should not write down both (L) and (M), but perform the cancelling on the line (L), which would then become (M): but a pupil would have been confused at first, if he had not seen both lines written in full.

CASE XI.

60. This case consists merely of a variety of *expressions* intended to exemplify the previous cases; and the working of the examples under it depends entirely upon a correct and ready use of the methods of performing Addition, Subtraction, &c., as already explained. There is no difficulty in any of them, so long as a pupil will pay close attention to the *signs;* but I will point out one or two Exs. wherein he is liable to err, though every such error must arise entirely from want of proper observation.

Ex. B 2, or B 5. In working these, a pupil must take care that he reduces the whole expression on each side of the (−) to the simplest form before he begins to perform the operation of sub^{n}, which that sign (−) bids him perform. The same caution would be necessary if there had been (+) instead of (−) in either of the above examples.

I will work B 5 as it ought to be written, merely changing the (—) into (+): and leave a large space on each side of the (+), so that I may run no risk of connecting the two parts incorrectly.

$$4\tfrac{4}{11} \text{ of } 7\tfrac{5}{16} \;+\; 2\tfrac{1}{2} \text{ of } 11$$

$$= \frac{\overset{3}{\cancel{48}}}{11} \times \frac{117}{\cancel{16}} \;+\; \frac{5}{2} \times \frac{11}{1}$$

$$= \frac{351}{11} \;+\; \frac{55}{2} \qquad \text{(L. C. D.} = 22)$$

$$= \frac{702 \;+\; 605}{22} = \frac{1307}{22} = 59\tfrac{9}{22}$$

Ex. C. 2. Simplify $\dfrac{3\frac{1}{2} \text{ of } \frac{8}{9}}{2\frac{2}{3} \text{ of } 3\frac{1}{2}} \div \dfrac{2\frac{3}{11} \text{ of } 1\frac{1}{2}}{3\frac{1}{3} \text{ of } 7\frac{1}{4}}$.

Now, the whole of the fr^{n} to the right of the sign ($\div$) is a divisor of the fr^{n} to the left of the sign; therefore this right-hand fr^{n} must be inverted, and the sign ($\div$) changed to ($\times$): the expression will then be

$$\frac{3\frac{1}{2} \text{ of } \frac{4}{7}}{2\frac{4}{7} \text{ of } 3\frac{1}{2}} \times \frac{3\frac{1}{2} \text{ of } 7\frac{4}{7}}{2\frac{3}{11} \text{ of } 1\frac{1}{12}};$$

and this, by changing *of* into $\times$, and the mixed numbers into impr fr^{ns}, becomes

$$\frac{\frac{7}{2} \times \frac{5}{8}}{\frac{17}{6} \times \frac{10}{3}} \times \frac{\frac{28}{9} \times \frac{54}{7}}{\frac{24}{11} \times \frac{11}{12}}$$

and lastly,—inverting the fr^{ns} in the lower line, because they are divisors, and connecting all the fr^{ns} with mult^{n} signs, I have

$$\text{the entire expression} = \frac{7}{2} \times \frac{5}{8} \times \frac{6}{17} \times \frac{3}{10} \times \frac{28}{9} \times \frac{54}{7} \times \frac{11}{24} \times \frac{12}{11} \quad \text{(N)}$$

$$= \frac{\overset{1}{7}}{\underset{1}{2}} \times \frac{\overset{1}{5}}{\underset{4}{8}} \times \frac{\overset{3}{6}}{17} \times \frac{\overset{1}{3}}{\underset{2}{10}} \times \frac{\overset{14}{28}}{9} \times \frac{\overset{6}{54}}{7} \times \frac{\overset{1}{11}}{\underset{2}{24}} \times \frac{\overset{1}{12}}{11}$$

$$= \frac{3 \times 3 \times 7 \times 3}{4 \times 17} = \frac{189}{68} = 2\frac{53}{68}.$$

In working such Exs. I should omit line (N) for the reason givin in (59).

In Exs. D 1, D 2, &c. the whole quantity enclosed in () parentheses, or brackets, as they are called, must be reduced to as simple a form as possible, before the $\times$ or $\div$ joining the expressions in brackets can be made use of.

Thus, D 1, to simplify $\left(\frac{7}{8} - \frac{3}{10}\right) \times \left(2\frac{4}{7} + 3\frac{1}{2}\right)$, I shall first reduce $\frac{7}{8} - \frac{3}{10}$ to the simplest form; then $2\frac{4}{7} + 3\frac{1}{2}$ must be reduced in like manner; and these two results must then be multiplied together, because the sign ($\times$) was placed between the brackets. The whole work would stand thus,

$$\frac{7}{8} - \frac{3}{10} = \frac{35 - 12}{40} = \frac{23}{40} \quad \text{I. (since L. C. D.} = 40)$$

and $2\frac{4}{7} + 3\frac{1}{2} = 2 + 3 + \frac{4}{7} + \frac{1}{2}$

$$= 5 + \frac{27 + 4}{36} = 5 + \frac{31}{36} = 5\frac{31}{36} \quad \text{II.}$$

Therefore, multiplying I. and II., I have

$$\left(\frac{7}{8}-\frac{3}{10}\right)\times\left(2\tfrac{1}{4}+3\tfrac{1}{5}\right)=\frac{23}{40}\times 5\tfrac{9}{20}$$

$$=\frac{23}{40}\times\frac{211}{36}$$

$$=\frac{4853}{1440}=3\tfrac{413}{1440}.$$

CASE XII.

DEF. Any quantity involving money, weight, &c., as 4 shillings, 5 pence, $\frac{3}{4}$oz., is called a *concrete* quantity; and any number not involving any such denominations,—as 6, 8, $\frac{4}{5}$, is called an *abstract* number.

61. To express in positive terms such quantities as $\frac{5}{8}$s., $\frac{1}{1}\frac{1}{2}$ of 10s. 6d. &c.; *i. e.* to express concrete quantities, being frac^{l} parts of any given den^{n}, in terms of lower den^{ns}.

Thus, $\frac{5}{8}$s. must be expressed in terms of pence, and frac^{l} parts of a penny; $\frac{1}{1}\frac{1}{2}$ of 10s. 6d. in terms of shillings, pence, and frac^{l} parts of 1d.

So, also, $\frac{9}{11}$ of a ton would be expressed in cwts., qrs., lbs., oz. drs., and frac^{l} parts of a dram.

It may happen that the proposed fr^{n} can be expressed in an exact number of units of some one of the lower den^{ns}; then, of course, there will be no frac^{l} part of the last named den^{n}.

62. I will now show how to perform the operation intended in the last article.

It has been seen (21) that $\frac{5}{8}$s. means either *one*-eighth of 5 shillings, or *five*-eighths of 1 shilling. We may, therefore, use either of these two methods:—divide the 5s. by 8,

as in Comp^{d} Short Division, and the process will be thus— $\begin{array}{r|l} & s.\ \ d. \\ 8) & 5\ \ 0 \\ \hline & 7\frac{1}{2}\text{d} \end{array}$ or, treating the fr^{n} as $\frac{5}{8}$ of 1s., we may change *of* into ($\times$), reduce the 1s. to pence, and proceed thus—

$$\frac{5}{8} \text{ of } 1\text{s.} = \frac{5\times 12}{8}\text{d.} = \frac{5\times \overset{3}{\cancel{12}}}{\underset{2}{\cancel{8}}}\text{d.} = \frac{15}{2}\text{d.} = 7\tfrac{1}{2}\text{d.}$$

This latter method seems more suitable to an Ex. which professes to belong to frac^{ns}, and is the one which I recommend, unless the den^{r} of the proposed fr^{n} be very large, so as to make the division to be performed rather difficult: in that case, to work by Comp^{d} Long Div^{n} is preferable, as being more likely to be correct. I will take such an Ex. and work it through by both methods.

Express in positive terms $\frac{17}{26}$ of a ton.

Working fractionally, we have

$$\frac{17}{26} \text{ of } 1 \text{ ton} = \frac{17\times \overset{10}{\cancel{20}}}{\underset{13}{\cancel{26}}}\text{cwts.} = \frac{170}{13}\text{cwts.} = 13\tfrac{1}{13}\text{ cwts.}$$

I now take the frac^{l} part of the cwts., viz. $\frac{1}{13}$ of 1 cwt., and express it in the next lower den^{n}, viz. quarters; and so proceed, till either there be no frac^{l} part left, or till I come to the last den^{n} used.

$$\frac{1}{13} \text{ of } 1 \text{ cwt.} = \frac{1\times 4}{13}\text{qrs.} = \frac{4}{13}\text{qrs. (}i.\ e.\text{ 0 qrs. in the final quotient.)}$$

$$\frac{4}{13} \text{ of } 1 \text{ qr.} = \frac{4\times 28}{13}\text{lbs.} = \frac{112}{13}\text{lbs.} = 8\tfrac{8}{13}\text{lbs.}$$

$$\frac{8}{13} \text{ of } 1 \text{ lb.} = \frac{8\times 16}{13}\text{oz.} = \frac{128}{13}\text{oz.} = 9\tfrac{11}{13}\text{oz.}$$

$$\frac{11}{13} \text{ of } 1 \text{ oz.} = \frac{11\times 16}{13}\text{drs.} = \frac{176}{13}\text{drs.} = 13\tfrac{7}{13}\text{drs.}$$

therefore, $\frac{17}{26}$ of 1 ton = cwt. 13 qrs. 0 lbs. 8 oz. 9 drs. $13\frac{7}{13}$.

Secondly, take $\frac{1}{26}$ of 17 tons; *i. e.* divide 17 tons by 26.

tons
17
20
26) 340 (13 cwt.
26
80
78
2
4
26) 8 (0 qrs.
28
26) 224 (8 lbs.
208
16
16
26) 256 (9 oz.
234
22
16
26) 352 ($13\frac{7}{13}$ drs.
26
92
78
14
$\frac{14}{26} = \frac{7}{13}$

therefore, as before, $\frac{17}{26}$ tons = cwt. 13 qrs. 0 lbs. 8 oz. 9 drs. $13\frac{7}{13}$

63. When Exs. under this Case involve money, it is well to carry the reduction no farther than pence and frac^{l} parts of a penny: for, since farthings are themselves written as frac^{l} parts of a penny, therefore if we carry the reduction to frac^{l} parts of a farthing, the answer will appear somewhat complicated. I will work an Ex. which will illustrate this.

Express in positive terms $\frac{151}{240}$sh.

$$\frac{151}{240}\text{s.} = \frac{151 \times \overset{1}{12}}{\underset{20}{240}}\text{d.} = \frac{151}{20}\text{d.} = 7\tfrac{11}{20}\text{d.} \quad (O)$$

$$\frac{11}{20}\text{d.} = \frac{11 \times \overset{1}{\cancel{4}}}{\underset{5}{\cancel{20}}}\text{far.} = \frac{11}{5}\text{f.} = 2\tfrac{1}{5}\text{f.}$$

therefore $\frac{151}{240}$s. $= 7\frac{1}{4}$d. $\frac{1}{5}$f. (P)

If I compare $7\frac{1}{10}$d. in (O) and $7\frac{1}{4}$d. $\frac{1}{5}$f. in (P), I see that the former fr^{n} involves less labour, and conveys to a person acquainted with fr^{ns} the idea of the real value of the expression, at least as well as the other.

64. When a unit of any den^{n} can be divided exactly by any number, as 4, 5, 6, &c., such fourths, fifths, sixths, &c., are called *aliquot* parts of that unit.

Thus, since £1 or 20s. when divided by 3, 4, 5, 6, 8, 10, 12, gives quotients involving an exact number of shillings and pence; therefore thirds, fourths, fifths, &c., of £1 are called aliquot parts of £1.

For example—

$\frac{1}{5}$ £ = 4s. $\frac{2}{5}$ £ = 8s. $\frac{4}{5}$ £ = 16s.

$\frac{1}{8}$ £ = 2s. 6d. $\frac{5}{8}$ £ = 12s. 6d. $\frac{7}{8}$ £ = 17s. 6d.

$\frac{1}{3}$ £ = 6s. 8d. $\frac{2}{3}$ £ = 13s. 4d.

So also, $\frac{1}{8}$sh. $= 1\frac{1}{2}$d. $\frac{5}{8}$sh. $= 7\frac{1}{2}$d. $\frac{7}{8}$sh. $= 10\frac{1}{2}$d.

And since 1 lb. avoirdupois, or 16oz. can be divided exactly by 2, 4, 8; therefore halves, fourths, and eighths of 1 lb. can be at once expressed in oz.

Thus, $\frac{3}{8}$lb. = 6oz. $\frac{7}{8}$lb. = 14oz.

and if the value of one or more of these aliquot parts of the denominations in common use be remembered, the operations performed in this Case will often be much shortened.

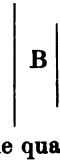
Ex. $\frac{27}{160}$£ $= \frac{27 \times \overset{1}{\cancel{20}}}{\underset{8}{\cancel{160}}}$sh. $= \frac{27}{8}$sh. $= 3\frac{3}{8}$sh. = 3s. $4\frac{1}{2}$d.

RATIO AND PROPORTION.

Before proceeding to Case XIII. it will be advisable to discuss one of the most important relations in numbers, without which a pupil could not understand the principle upon which Exs. in this Case are worked. This relation is termed RATIO.

And as this consideration of Ratio leads to the doctrine of PROPORTION, I have thought it well to continue the subject in one unbroken thread, and then refer to this explanation, when I come to the treatment of questions involving Proportion and its applications.

65. When two quantities are placed before us—as, for instance, the lines A, B—and we wish to compare their magnitude, we perceive that there is a certain relation between them, which we familiarly call the comparative size of them; and this comparison or relation is measured by seeing how often the one quantity contains the other.



Now, to see how often one quan^{y} contains the other—as, for example, how often 36 contains 5—we divide the former by the latter, *i. e.* the 36 by the 5, and the quotient is $\frac{36}{5}$; therefore, to see how often the line A contains the line B, I divide A by B: *i. e.* the relation of A to B is expressed by the fr^{n} $\frac{A}{B}$.

66. This relation is called the *ratio* of A to B, and is thus expressed, A : B.

If A > B, the fr^{n} expressing the ratio of A to B, *i. e.* $\frac{A}{B}$, will be an improper fr^{n} greater than 1;

If A = B, the fr^{n} will become $\frac{A}{A}$, or unity;

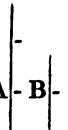
If A < B, the fr^{n} will become $\frac{A}{B}$, a proper fr^{n}.

Also, when the fr^{n} $\frac{A}{B}$ is greater than 1, the ratio of A to B is called a ratio of *greater* inequality, because A > B.

When the fr^{n} $\frac{A}{B} = \frac{A}{A} = 1$, the ratio is called a ratio of equality; and when the fr^{n} $\frac{A}{B}$ is a proper fr^{n}, the ratio is called a ratio of *less* inequality, because A < B.

67. If the num^{r} and den^{r} of a fr^{n} expressing ratio both consist of the same kind of *concrete* quantities, the ratio between these quantities will be an *abstract* number.

Ex. If the lines A and B represent 4 inches and 3 inches respectively, then $A : B = \frac{A}{B} = \frac{4 \text{ in.}}{3 \text{ in.}} = \frac{4}{3}$; and if, instead of inches, these lines had represented 4 feet and 3 feet, or 4 lb. and 3 lb., the ratio of A to B would still be $\frac{4}{3}$, and of B to A would be $\frac{3}{4}$.



But if the num^{r} and den^{r} be of the same species of concrete quantity, but not expressed in the same denomination, the ratio cannot be represented by an abstract quantity, until they both be reduced to the same denomination. Thus, $\frac{4 \text{ feet}}{9 \text{ inches}}$ does not $= \frac{4}{9}$ feet, or $\frac{4}{9}$ in. or $\frac{4}{9}$, but it $= \frac{48 \text{ in.}}{9 \text{ in.}} = \frac{48}{9} = \frac{16}{3}$, an abstract number.

If quantities be not of the same nature, there can be no ratio between them; thus $\frac{4 \text{ feet}}{5 \text{ minutes}}$ is no ratio at all, since we cannot compare the magnitude of 4 feet and of 5 minutes.

This article must be especially noticed, as upon it will depend the mode of working Exs. in Case XIII.

68. We now see that any fraction, as $\frac{2}{5}$, has still a further meaning, beside what was stated in (21), viz. *the ratio of* 2 *to* 5, or, as it is written, 2 : 5; and $\frac{2}{5}$ may be called *a ratio.*

If, then, two other quantities be taken, as 4 and 10, and it is found that $\frac{4}{10} = \frac{2}{5}$, then we say that the ratio of 4 to 10 is equal to the ratio of 2 to 5, and this fact we express either thus,

$$4 : 10 = 2 : 5, \text{ or } 4 : 10 :: 2 : 5;$$

the latter expression is thus read—4 is to 10 as 2 is to 5.

When, therefore, two ratios, as $\frac{4}{10}$ and $\frac{2}{5}$, are placed before us, and we learn that they are equal, we say that the four quantities, 4, 10, 2, 5, are *Proportionals.*

69. We may now define Proportion to consist in the equality of two ratios. Of the above-mentioned numbers 4, 10, 2, 5—4 and 5 are called the *extremes;* 10 and 2 are called the *means,* because they are intermediate between the extremes.

Also, when 4 quantities are given, and we wish to ascertain whether they are proportionals or not, we must see if the fr^{n} expressing the ratio of the 1st and 2nd = the fr^{n} expressing the ratio of the 3rd and 4th. Thus, if I take the numbers 5, 6, 7, 8, and wish to try whether they are proportionals, I compare $\frac{5}{6}$, the ratio of the first pair, and $\frac{7}{8}$, the ratio of the second pair; and if these fr^{ns} are proved unequal, the above 4 quantities are not proportionals.

Now, it was shown in (50) that to compare two frac^{l} quantities we must bring them to some Com. Den^{r}. Reducing the fr^{ns} $\frac{5}{6}$ and $\frac{7}{8}$ to a c.d. 48, we have to compare

$$\frac{5 \times 8}{48} \text{ and } \frac{6 \times 7}{48} \qquad (Q)$$

And since $\frac{40}{48}$ is not $= \frac{42}{48}$, therefore $\frac{5}{6}$ is not $= \frac{7}{8}$, and 5, 6, 7, 8, are not proportionals.

It will be seen in line (Q) that the test as to whether or not the four proposed quantities be proportionals, consists in multiplying, 1st, the two extremes 5, 8, and 2ndly, the two means 6, 7: and if these products be equal, the four quantities are proportionals; if they be not equal, the four quantities are not proportionals.

This operation may generally be performed *mentally:* thus, if I take 5, 8, 9, 16, I say $5 \times 16 = 80$, and $8 \times 9 = 72$; therefore these quantities are not proportionals. Again, if I take 4, 9, 5, $11\frac{1}{4}$, since the product of the extremes $4 \times 11\frac{1}{4} = 4 \times \frac{45}{4} = 45$, and the product of the means $9 \times 5 = 45$; therefore these last four quantities are proportionals. (See Appendix, Art. Ratio).

70. Having now explained how to tell when four quantities are proportionals, we proceed to show how, when *three* numbers are given, we may find a fourth number, such that the other three and this fourth shall be proportionals. This is sometimes called finding a fourth proportional to three given terms. But, in order to explain this process the more readily, I will first explain some operations in Fractions which have not yet been noticed.

71. We have seen (28, 29) that to multiply a fr^{n} by any number, we may either multiply the num^{r} or divide the den^{r} by this multiplier. Thus, if I wish to multiply $\frac{4}{10}$ by 10, it becomes

$$\frac{4 \times 10}{10} = \frac{4 \times \overset{1}{\cancel{10}}}{\underset{1}{\cancel{10}}} = 4;$$

or, $$\frac{4}{10 \div 10} = \frac{4}{1} = 4:$$

therefore, if I wish to multiply a fr^{n} by its own den^{r}, I obtain the correct result by merely taking away that den^{r}.

Now, if I multiply two equal quantities by the same multiplier, the products will be equal. If, therefore, in the equation,

$$\frac{4}{10} = \frac{2}{5} \qquad \text{(I.)}$$

I multiply both sides by 10; the first side becomes 4, as just shown, and the second side becomes

$$\frac{2 \times 10}{5}; \text{ therefore } 4 = \frac{2 \times 10}{5} \qquad \text{(II.)}$$

By comparing (I.) and (II.) I find that I have transferred a den^{r} of one side of the equation into the num^{r} of the other, without disturbing the equality.

Again, it has been shown (9) that if I have a product, as $7 \times 5 \times 3$, and I wish to divide this product by one or more of its factors, I have merely to strike out such one or more factors; *i. e.* if I wish to divide $7 \times 5 \times 3$ by 5, I remove the 5, and have as quotient 7×3; therefore, if in equation (II.) I wish to divide the right-hand side by 2, I have only to cancel the 2, and leave as the quotient $\frac{10}{5}$: also, dividing the left-hand side by 2, by making 2 as the den^{r} (22), but *not performing* the division, I have the quotient $\frac{4}{2}$.

Since, then, I have divided both sides of the equation (II.) by 2, the quotients must be equal;

$$\textit{i. e.} \ \frac{4}{2} = \frac{10}{5} \qquad \text{(III.)}$$

On comparing (I.) and (III.) it will be seen that I have now transferred a factor 2 from the num^{r} of the right side to the den^{r} of the left; or made a multiplier of one side *into a divisor of* the other: also, that I have transferred a

quantity 10 from the den^{r} of the left side to the num^{r} of the right side. And these processes might have been performed, whatever had been the two fractions given in (1.)

72. We may now, therefore, conclude, that if two equal fractions are placed before us, we can transfer any factor from the num^{r} of one side to the den^{r} of the other, and from the den^{r} of one side to the num^{r} of the other, and the equality will still exist. But it must be distinctly noticed, that we do not change the places of these factors by subtraction and addition, but by division and multiplication: thus, if I have

$$\frac{3}{5} = \frac{6}{10} \text{ or } = \frac{2\times3}{10} \qquad \text{(IV.)}$$

and I wish to remove a 2 from the right-hand num^{r} to the left-hand den^{r}, I *divide* the num^{r} by the 2, and *multiply* the other den^{r} by it, and I have

$$\frac{3}{5\times2} = \frac{1\times3}{10}:$$

now, it is not necessary to write this fig. 1 as I have done, because $1\times3=3$; but let us remove the 3 from the right-hand num^{r}, and I have

$$\frac{3}{5\times2\times3} = \frac{1\times1}{10} = \frac{1}{10}.$$

It appears, then, that if I remove *all* the factors, *i. e.* the whole num^{r}, I must remember that I leave as many quotients 1 as I removed factors, and therefore the num^{r} or den^{r} from which the factors have been taken = the product of all these figs. 1, and is therefore = 1. If this product, which = 1, occurs in the den^{r}, we do not write it down, because $\frac{6}{1} = 6$.

I will now make use of this power of changing the places of num^{r} and den^{r}, as just explained.

We know that $\frac{3}{7} = \frac{15}{35}$; (I.)

Change, first the 7, and second by the 15, and we have

$$3 = \frac{7 \times 15}{35} \text{ (II.)} \quad \text{and} \quad \frac{3}{15} = \frac{7}{35} \text{ (III.)}$$

Next, in III., change first the 3, and second the 35, and we have

$$\frac{1}{15} = \frac{7}{3 \times 35} \text{ (IV.)} \quad \text{and} \quad \frac{35}{15} = \frac{7}{3} \text{ (V.)}$$

Comparing I. and V., I find that V. is I. inverted: hence, I learn that if two frⁿˢ are equal, they will be equal when inverted. However complicated the two equal fractions may be, this will make no difference:

Thus, if $\frac{3\frac{1}{2}}{7\frac{1}{8}} = \frac{35}{72}$,

we can say, $3\frac{1}{2} = \frac{7\frac{1}{8} \times 35}{72}$, or $72 = \frac{7\frac{1}{8} \times 35}{3\frac{1}{2}}$, or $\frac{72}{35} = \frac{7\frac{1}{8}}{3\frac{1}{2}}$.

73. Having established the truth of the above operations, I proceed to solve the following question.

If three numbers be given, as 5, 6, 10, what number must be taken as a fourth number, such that the four, when taken in order, shall be proportionals?

Now, as I do not know (or, at least, the learner does not know,) the required number, let the letter N stand for or represent this number; and I have now to try and find what this N must be.

Since, then, 5, 6, 10, and N are required to be proportionals, therefore we must have the ratio between the first pair 5, 6 = that between the second pair 10, N; *i. e.*

$$\frac{5}{6} = \frac{10}{N}; \qquad (R)$$

change 5 and N, as shown above, and we have $\frac{N}{6} = \frac{10}{5}$. Next, throw the 6 into the right-hand num^{r}, and we find

$$N = \frac{6 \times 10}{5} \qquad (S)$$

$$= \frac{6 \times \overset{2}{\cancel{10}}}{\underset{1}{\cancel{5}}} = 12.$$

Put 12 for N in (R), and we have $\frac{5}{6} = \frac{10}{12}$, which we know to be true, and therefore 5, 6, 10, 12, are proportionals.

The result of this article must be most carefully noticed; for in observing (S) we learn that the required 4th number N was thus formed from the three former ones, 5, 6, 10: viz. that it is the value of this fr^{n}—the product of the 2nd and 3rd terms, divided by the first; or it $= \frac{2\text{nd} \times 3\text{rd}}{1\text{st}}$.

74. We have shown that, in order that there may exist a ratio between two quantities, they must be of the same kind; since, then, there is to be a proportion among the three given quantities and the fourth, therefore between the 1st and 2nd there must be a ratio, and an equal ratio between the 3rd and 4th. Hence we see that the second pair must be of the same kind; that is, the 4th quantity, when found, ought to be of the same kind as the 3rd. Now, if we observe the fr^{n} $\frac{2\text{nd} \times 3\text{rd}}{1\text{st}}$ in the last article, we notice, that since it is allowed that the 2nd and 1st are alike in kind, therefore the $\frac{2\text{nd}}{1\text{st}}$ will be an abstract number (67); and therefore the 4th, which $= \frac{2\text{nd}}{1\text{st}} \times 3\text{rd}$, $= 3\text{rd}$ multiplied by some abstract number; *i. e.* it will be the 3rd term repeated as many times as there are units in this

number, and therefore will be of the same kind as that in which the third term was expressed.

Also, if the 2nd term be greater than the 1st, $\frac{2nd}{1st}$ will be a fr^n > 1, and the multiplier of the 3rd term will be greater than 1, and the 4th term greater than the 3rd; but if the 2nd term be less than the 1st, then $\frac{2nd}{1st}$ will be a proper fr^n, and the multiplier less than 1, and the 4th term less than the 3rd.

75. We may here notice a form of expression that we familiarly use, when we say, that two quantities are proportional to one another; as, for instance, that the amount of a servant's wages is proportional to the length of his service. Here, apparently, two quantities of different kinds are compared, viz. money and time; but, in reality, four quantities are implied, viz. two periods of time, and two amounts of wages. For since one year and one year's wages may be taken as fixed standards, by which we may measure other periods and other amounts; therefore, I mentally compare any proposed length of service and the corresponding amount of wages with these fixed standards; and the original expression means this—that there is the same ratio between any given length of service and one year, that there is between the amount of wages for that service and one year's wages; or, that these two periods of service and two amounts of wages together form a proportion in this order,

Given time : 1 year :: wages for that time : wages for 1 year.

If we wish to ascertain whether two quantities are what is here called proportional to one another, according to the common usage of the words, we may try if doubling

the one quantity causes the other to be doubled; as for example, in this instance—will the wages be doubled, if the time of service be doubled? Since this is the case, therefore the wages and time are proportional in the sense explained above.

The correct, though not common mode of expressing this proportion, is to say, that the amount of wages *varies as* the length of the time of service.

76. But sometimes this proportion occurs in a different form; as, when a man has a certain number of miles to walk in any time; then, if he quickens his rate of going, he can complete his task in less time. In this case it would be incorrect to say that the length of time was in proportion to the rate of walking; for, according to the test given above, I ask—if the rate be doubled, will the number of days be *doubled?* the answer is, no: on the contrary, it will be *halved.* There is here, then, evidently a proportion (taking the same word in the common usage,) but of a different kind, viz., that as one number is increased by multiplication, the other is correspondingly diminished by division.

As, in the first instance, it was said that the wages *varied as* the time, so, in this second example, the relation between the two quantities is correctly expressed by saying, that the number of days *varies inversely as* the rate of walking; or, as it is commonly said—in the first Ex., the wages were *directly* proportional to the time; and in the second Ex. the number of days is *inversely* proportional to the rate.

CASE XIII.

77. We are now able to perform the operation required under this Case, viz., to express one quantity in terms of another, or as a fracl part of another. This frn connecting the two quantities will show the ratio between them, and may be either proper or improper.

For example—to express 9d. in terms of £1, is to see what ratio 9d. bears to £1. Hence, according to what has been explained concerning ratio, we say,

$$\text{9d. in terms of £1} = \text{9d.} : \text{£1}$$
$$= \frac{\text{9d.}}{\text{£1}} = \frac{\text{9d.}}{\text{240d.}} = \frac{9}{240} = \frac{3}{80};$$

and transferring the £1 from the denr of the left-hand frn to the numr of the right-hand frn by (71), I have 9d. $= \frac{3}{80} \times$ £1, or $\frac{3}{80}$ of £1. This result may also be obtained as follows:—

I have to express pence in terms of £1.

Now, 240 pence = £1; therefore, taking $\frac{1}{240}$th part of both sides—

$$1 \text{ penny} = \frac{1}{240} \text{ of £1};$$

and multiplying by 9,

$$9 \text{ pence} = \frac{9}{240} \text{ of £1}$$
$$= \frac{3}{80} \text{ of £1};$$

and transferring the £1 from the right-hand numerator to the left-hand denominator, we have, as before,

$$\frac{9 \text{ pence}}{\text{£1}} = \frac{3}{80}.$$

It will be observed, that by this second method we have an independent proof, that the ratio between two concrete numbers of like kind, as pence, pounds, &c., is an abstract number.

78. Either of the two methods just shown may be employed; but I prefer the former, because it more decidedly keeps the idea of ratio before the mind; and an Ex. so worked can be written out in a more condensed from than by the latter method. It will be noticed that the two concrete quantities are *both* to be reduced to some com. den^{n}, in order that the ratio between them may be expressed: thus, in the last Ex., both were reduced to pence.

Ex. II. Express 7s. 7½d. in terms of 10s. 6d.

Here I exhibit both numerator and denominator in terms of pence: and write

$$\frac{7\text{s. }7\tfrac{1}{2}\text{d.}}{10\text{s. }6\text{d.}} = \frac{91\tfrac{1}{2}\text{d.}}{126\text{d.}} = \frac{\frac{183}{2}}{126} = \frac{183}{2\times 126} = \frac{61}{84}.$$

I might have reduced both numerator and denominator to half-pence, but a pupil would not easily reduce 7s. 7½d. to half-pence mentally; but he might write it in pence and a fraction of a penny, as I have done, and then the successive steps of the work will be seen more clearly.

Referring to Ex. 1., we read— $\frac{9\text{d.}}{£1} = \frac{3}{80}$; and since, by (72), two equal fractions may be inverted, without disturbing the equality; therefore $\frac{£1}{9\text{d.}} = \frac{80}{3}$; or, changing the 9d. to the right-hand numerator,

$$£1 = \frac{80}{3} \times 9\text{d.}; \text{ or } = \frac{80}{3} \text{ of } 9\text{d.}$$

Hence we see that in every Ex. where we have to express one quantity in terms of a second, we can express this second in terms of the first, by merely inverting the ratio which connected the first and second. Thus, in Ex. II.,

$$\text{since } 7\text{s. }7\tfrac{1}{2}\text{d.} = \frac{61}{84} \text{ of } 10\text{s. }6\text{d.};$$

$$\text{therefore } 10\text{s. }6\text{d.} = \frac{84}{61} \text{ of } 7\text{s. }7\tfrac{1}{2}\text{d.}$$

79. Sometimes fractions may be involved in both num^{r} and den^{r}, as in

Ex. III. Express $1\frac{3}{4}$ of 2a. 1r. in terms of 3 acres $2\frac{1}{4}$ roods.

Here, bringing both quantities into roods, and changing *of* into ×, we have

$$\frac{1\frac{1}{2}\text{ of }2\text{a. }1\text{r.}}{3\text{a. }2\frac{1}{2}\text{r.}} = \frac{\frac{1}{2}\times 9\text{r.}}{14\frac{1}{2}\text{r.}} = \frac{\frac{1}{2}\times 9}{\frac{29}{2}}$$

$$= \frac{7}{\frac{1}{2}} \times \frac{9}{1} \times \frac{2}{29} = \frac{63}{58}.$$

This should be left as an improper fraction, because I can then read off this desired result—that

$$1\frac{1}{2}\text{ of }2\text{a. }1\text{r. : }3\text{a. }2\frac{1}{2}\text{r.} = 63 : 58;$$

or that the ratio of these two portions of land is that of 63 to 58.

As in (64) it was shown to be advisable to be able to express aliquot parts of any den^{n} in terms of lower den^{ns}, so, with a little experience, a pupil will, by a reverse process, mentally work easy Exs. in this Case, so as at once to see, that 5s. in terms of £1 = $\frac{1}{4}$; that 6s. 8d. : £1 = $\frac{1}{3}$, and so on: and a readiness in performing such simple reductions will often materially shorten the labour of more complicated Exs. Thus, to express $\frac{7}{8}$ of 13s. 4d. in terms of 10s.:—

By observing that 13s. 4d. = $\frac{2}{3}$£, and 10s. = $\frac{1}{2}$£, I have

$$\frac{\frac{7}{8}\text{ of }13\text{s. }4\text{d.}}{10\text{s.}} = \frac{\frac{7}{8}\text{ of }\frac{2}{3}\text{£}}{\frac{1}{2}\text{£}} = \frac{7}{\frac{4}{2}} \times \frac{2}{3} \times \frac{2}{1} = \frac{7}{6};$$

or, $\frac{7}{8}$ of 13s. 4d. = $\frac{7}{6}$ of 10s.

By the ordinary method I should have reduced both 13s. 4d. and 10s. to pence or fourpences, and the resulting fraction would have required much heavier reduction than mine has needed.

80. When either the num^{r} or den^{r} of the left-hand fr^{n} requires much reduction to a lower den^{n}, it is better to express by signs the multiplication requisite to perform this reduction, rather than to perform the mult^{n}, and put down the result; because, when either num^{r} or den^{r} is thus

expressed in factors, it is in the best form for detecting the probability of any cancelling taking place, so as to present the resulting fr^{n} in its lowest terms.

For example, in reducing 5 drs. 1 sc. 15 grs. to the fraction of 1 lb., I must bring the 1 lb. to grains, and I write

$$\frac{5\,\text{dr. } 1\,\text{sc. } 15\,\text{grs.}}{1\,\text{lb.}} = \frac{16\,\text{sc. } 15\,\text{grs.}}{1\times12\times8\times3\,\text{sc.}} = \frac{335\,\text{grs.}}{12\times8\times3\times20\,\text{grs.}} \quad \text{(T)}$$

$$= \frac{\overset{67}{\cancel{335}}}{12\times8\times3\times\underset{4}{\cancel{20}}} = \frac{67}{96\times12} = \frac{67}{1152}.$$

But in this Ex. we may also notice that the last denomination 15 grs. may be very readily expressed as a fraction of the preceding denomination, viz. scruples, for 15 grs. $= \frac{15}{60}$ sc. $= \frac{1}{4}$ sc.; therefore instead of the two latter fractions in line (T), I should write

$$= \frac{16\frac{1}{4}\,\text{sc.}}{12\times8\times3\,\text{sc.}} = \frac{\frac{67}{4}}{12\times8\times3} = \frac{67}{4\times12\times8\times3} = \frac{67}{12\times96}$$

$$= \frac{67}{1152}.$$

Of these five fractions the last four are merely reductions to a simpler form; and the number of steps which a pupil may have to write down in working Exs. similar to the one above will depend, partly on the magnitude of the numbers involved, and partly on his own quickness in working mentally.

I will give one more Ex. in which the work is condensed, but still the successive operations are intelligible.

Express $\frac{11}{12}$ of $4\frac{1}{2}$d. in terms of 9s. $7\frac{1}{2}$d.

$$\frac{\frac{11}{12}\text{ of }4\frac{1}{2}\text{d.}}{9\text{s. }7\frac{1}{2}\text{d.}} = \frac{\frac{11}{12}\times4\frac{1}{2}\text{d.}}{115\frac{1}{2}\text{d.}} = \frac{\overset{11}{\cancel{11}}}{\underset{1}{\cancel{12}}}\times\frac{17}{\underset{2}{\cancel{4}}}\times\frac{\overset{1}{\cancel{2}}}{\underset{21}{\cancel{231}}} = \frac{17}{504};$$

$$\text{or, } \frac{11}{12}\text{ of }4\tfrac{1}{2}\text{d.} = \frac{17}{504}\text{ of }9\text{s. }7\tfrac{1}{2}\text{d.}$$

CASE XIV.

81. Under this head are arranged—1st. expressions involving Exs. of Case XII., and requiring both Addition and Subtraction, as—

I. $\frac{5}{8}$£ + $\frac{3}{4}$ of 21s. − $\frac{7}{8}$ of 27s.

2ndly, Exs. in Compound Addition, Subtraction, Multiplication, and Division, in which fractional quantities are involved. I give one of each.

II. £3 6s. 8$\frac{1}{2}$d. + £2 11s. 8$\frac{1}{2}$d. + £4 13s. 9$\frac{1}{2}$d.

III. £7 16s. 8$\frac{1}{2}$d. − £3 11s. 10$\frac{1}{2}$d.

IV. £8 14s. 2$\frac{7}{12}$d. × 46$\frac{1}{2}$d.

V. £38 3s. 4$\frac{1}{2}$d. ÷ 13$\frac{1}{2}$.

82. In Ex. I. we may use either of the following methods, viz. reduce all the quantities to the same denomination, and then find the value of their sum by Case XII.; or express in positive terms each fraction separately, and then find the value of the whole by Compound Addition and Subtraction.

By the first method, reducing the quantities to the fraction of £1, I have

$$\frac{5}{8}\text{ sov.} + \frac{3}{4}\text{ of 21s.} - \frac{7}{8}\text{ of 27s.}$$

$$= \frac{5}{8}£ + \frac{3}{4} \times \frac{21}{20}£ - \frac{7}{8} \times \frac{27}{20}£$$

$$= \left(\frac{5}{8} + \frac{63}{80} - \frac{189}{160}\right)£$$

$$= \frac{100 + 126 - 189}{160}£ \qquad (\text{since L. C. D.} = 160)$$

$$= \frac{226 - 189}{160}£ = \frac{37}{160}£$$

and $\frac{37}{160}$ £ $= \frac{37 \times 20}{160}$ sh. $= \frac{37}{8}$ sh. $= 4\frac{5}{8}$ sh. $=$ 4s. 7½d.

By the second method—

$\frac{5}{8}$ £ (64) $=$ 12s. 6d.

$\frac{3}{4}$ of 21sh. $= \frac{63}{4}$ sh. $= 15\frac{3}{4}$ sh. $=$ 15s. 9d.

$\frac{7}{8}$ of 27sh. $= \frac{189}{8}$ sh. $= 23\frac{5}{8}$ sh. $=$ £1 3s. 7½d.

therefore $\frac{5}{8}$ £ + $\frac{3}{4}$ of 21s. − $\frac{7}{8}$ of 27s. $=$ 12s. 6d. + 15s. 9d. − £1 3s. 7½d.

$=$ £1 8s. 3d. − £1 3s. 7½d.

$=$ 4s. 7½d.

83. Ex. II. £3 6s. 8½d. + £2 11s. 8½d. + £4 13s. 9½d.

Commencing with the addition of the fractional parts, I have

$\frac{5}{6}$d. + $\frac{7}{8}$d. + $\frac{2}{3}$d. $= \frac{20 + 21 + 16}{24}$ d.

$= \frac{57}{24}$d. $= \frac{19}{8}$d. $= 2\frac{3}{8}$d.

and carrying the 2d. to the row of pence, I complete the sum as in Com pound Addition; and the whole amount $=$ £10 12s. 3⅜d.

84. Ex. III. £7 16s. 8½d. − £3 11s. 10½d.

Commencing with the subtraction of the fractional parts, I observe that $\frac{5}{6} < \frac{7}{8}$; therefore, borrowing 1d. from the 8d. in the former quantity, I have

$1\frac{5}{6}$d. − $\frac{7}{8}$d. $= \left(\frac{11}{6} - \frac{7}{8}\right)$d. $= \frac{44 - 21}{24}$d. $= \frac{23}{24}$d.

and completing the sum by Comp^{d} Sub^{n}, the whole difference is £4 4s. 9$\frac{23}{24}$d.

Ex. IV. £8 14s. 2$\frac{1}{12}$d. × 46$\frac{2}{3}$.

Now, since to multiply 46$\frac{2}{3}$ is to repeat the multiplicand 46 times and $\frac{2}{3}$ times, therefore, if I multiply the given

comp^{d} quant^{y} by 46, and then take $\frac{8}{9}$ of the same, the sum of the two products will be the product required, viz. $46\frac{8}{9}$ times £8 14s. $2\frac{7}{12}$d. Proceeding as in Comp^{d} Mult^{n}, and working in the margin on the paper those parts of the mult^{n} which involve fractions, I have

£8 14s. $2\frac{7}{12}$d.
$9\times5+1=46$

£78 7s. $11\frac{1}{4}$
5

£391 19s. $8\frac{1}{4}$ = 45 times
£8 14s. $2\frac{7}{12}$ = 1 „
£7 14s. $10\frac{8}{9}\frac{7}{12}$ = $\frac{8}{9}$ „
£408 8s. $9\frac{7}{12}$ = $46\frac{8}{9}$ „

$\left[\overset{3}{\cancel{8}}\times\frac{7}{\cancel{12}_{4}}\text{d.}=\frac{21}{4}\text{d.}=5\frac{1}{4}\text{d.}\right]$

And $\frac{8}{9}$ of (£8 14s. $2\frac{7}{12}$d.)

$=\frac{8\times(\text{£8 14s. }2\frac{7}{12}\text{d.})}{9}$

$=\frac{\text{£69 13s. }8\frac{1}{2}\text{d.}}{9}$

$\left[\text{for }\overset{2}{\cancel{8}}\times\frac{7}{\cancel{12}_{3}}\text{d.}=\frac{14}{3}\text{d.}=4\frac{2}{3}\text{d.}\right]$

= £7 14s. $10\frac{8}{9}\frac{7}{12}$d.

The $\frac{8}{9}\frac{7}{12}$d. is thus obtained: after dividing the pence by 9, there remained 2d.; therefore I had to divide $2\frac{1}{2}$d. by 9—the quotient

$$=\frac{2\frac{1}{2}\text{d.}}{9}=\frac{\frac{8}{3}}{9}\text{d.}=\frac{8}{27}\text{d.}$$

Sum of the fractions $=\left(\frac{1}{4}+\frac{7}{12}+\frac{8}{27}\right)\text{d.}=\frac{27+63+32}{108}\text{ d.}$

$$=\frac{122}{108}\text{d.}=\frac{61}{54}\text{ d.}$$

$$=1\frac{7}{54}\text{d.}$$

Again, $46\frac{8}{9}=\frac{422}{9}$; therefore I may multiply the given sum by $\frac{422}{9}$; *i. e.* multiply by 422, and divide by 9. But as this second method involves more labour, and would therefore be not so generally used, it is not worth while to work the Ex. out, especially as the process of Compound Multi-

plication, where fractions are involved, is sufficiently shown in the former method.

85. Ex. V. £38 3s. $4\frac{1}{2}$d $\div$ $13\frac{7}{8}$.

There is but one mode of working this Ex. viz. to reduce the divisor to an imp^{r} fr^{n}—invert it, and proceed as in multiplication.

Now, $13\frac{7}{8} = \frac{124}{9}$; and this, when inverted, becomes $\frac{9}{124}$. Therefore the Ex. becomes £38 3s. $4\frac{1}{2}$d. $\times \frac{9}{124}$; *i. e.* I have to multiply the given sum by 9, and divide by 124. The work will be as follows :—

	£	s.	d.	
	38	3	$4\frac{1}{2}$	
			9	$9 \times \frac{1}{2}$d. $= \frac{9}{2}$d. $= 5\frac{1}{2}$d.
124)	343	10	$5\frac{1}{2}$	(£2
	248			
	95			
	20			
124)	1910			(15s.
	124			
	670			
	620			
	50			
	12			
124)	605			(4d.
	496			
	109			

There now remain 109d. and $\frac{2}{5}$d., which have not yet been divided by 124; the quotient =

$$\frac{109\frac{2}{5}\text{d.}}{124} = \frac{\frac{547}{5}}{124}\text{d.} = \frac{547}{5 \times 124}\text{d.} = \frac{547}{620}\text{d.}$$

and the whole quotient is £2 15s. $4\frac{11}{12}\frac{7}{0}$d.

MISCELLANEOUS EXAMPLES.

86. These, of course, consist of questions involving one or more of the Cases which we have considered; and all that can be done towards guiding the pupil in working such Exs. *is*, merely to assist him in judging

what operations are required in solving any proposed question.

What was said in Case XI. applies more particularly here, viz., that it is necessary most carefully to note the *signs*, or rather the words which express signs. As an example of this caution, I will consider A 4; and in order to work it, I shall endeavour to see what operations are intended by the words employed in the question. The sum is as follows:

Multiply the sum of $\frac{2}{3}$, $\frac{3}{4}$ of $\frac{5}{6}$, and 4, by $7\frac{1}{1\frac{1}{2}}$.

Now, *the sum of* means addition, or (+), and *of* means (×); therefore, this Ex. expressed in signs is $(\frac{2}{3}+\frac{3}{4}\times\frac{5}{6}+4) \times 7\frac{1}{1\frac{1}{2}}$. This is now in the form of an Ex. under Case XI., and the remaining work may be completed by a pupil.

87. B 5. How many persons may receive each $3\frac{1}{3}$s. out of £$13\frac{1}{3}$?

This is only a Reduction sum, involving fractions; and, in plainer language, means—how often are $3\frac{1}{3}$s. contained in £$13\frac{1}{3}$? or, if I divide £$13\frac{1}{3}$ by $3\frac{1}{3}$s., what is the quotient?

The quotient required is $\frac{£13\frac{1}{3}}{3\frac{1}{3}\text{s.}}$

$$= \frac{13\frac{1}{3}\times 20\text{s.}}{3\frac{1}{3}\text{s.}} = \frac{\frac{69}{5}\times\frac{20\text{s.}}{1}}{\frac{23}{6}\text{s.}} \qquad \text{(U)}$$

$$= \frac{\overset{3}{\cancel{69}}}{\underset{1}{\cancel{5}}}\times\frac{\overset{4}{\cancel{20}}}{1}\times\frac{6}{\underset{1}{\cancel{23}}} = 72$$

By (67) we know that the second fraction in line (U) will be an abstract number; and the result, 72, shows that $3\frac{1}{3}$s. are contained 72 times in £$13\frac{1}{3}$.

88. E 6. Compare $\frac{3}{7}$£, $\frac{3}{8}$ of a guinea, and $\frac{2}{3}$ of 11s. $10\frac{1}{2}$d.

We cannot compare quantities without bringing them to some common name; *i. e.* in order to compare fractional quantities, we must reduce them to their L. C. D.

These quantities may therefore either be expressed in positive terms, as pounds, shillings, pence, &c., and then compared; or may be all expressed as fractions of one common quantity—say, of £1, or of 1 guinea, and then reduced to L. C. D. I use the latter method, because then I see more clearly the ratio between the quantities.

89. For the solution of such Exs. as A 3, F 9, G 4, see Arts. *Area* and *Volume*.

I will work in full another Ex., B 6, because upon the method of working it depends the solution of many questions in Arithmetic and Algebra.

Ex. $\frac{1}{3}+\frac{1}{4}+\frac{1}{12}$ of a number is 36, what is the number?

Now $\frac{1}{3}+\frac{1}{4}+\frac{1}{12}=\frac{4+3+1}{12}=\frac{8}{12}=\frac{2}{3}$.

And, by the question, this sum $=36$;
i. e. $\frac{2}{3}$ of the number $=36$;
therefore, dividing both sides by 2,
$\frac{1}{3}$ of the number $=18$;
and, multiplying by 3,
$\frac{3}{3}$ of the number, *i. e.* the whole number $=3\times18=54$.

* 90. The following is an Ex. which is to be worked upon a principle similar to the last.

If A can do a piece of work in 3 hours, B in 5 hours, and C in 7 hours, in what time can they do it, all working together?

Now, A can do the work in 3 hours;
therefore A can do $\frac{1}{3}$ of the work in 1 hour;
so, B „ $\frac{1}{5}$ „ 1 hour;
and C „ $\frac{1}{7}$ „ 1 hour;
therefore the three, A, B, C, working together, can perform

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7} \text{ or, } \frac{35+21+15}{105} \text{ or } \frac{71}{105} \text{ in 1 hour;}$$

therefore, if they do $\frac{71}{105}$ in 1 hour,

they will do $\frac{1}{105}$ in $\frac{1}{71}$ hour,

and therefore $\frac{105}{105}$, or the whole, in $\frac{105}{71}$ hours;

that is, in $1\frac{34}{71}$ hours.

By observing the fr^{n} $\frac{71}{105}$, expressing the amount done in 1 hour, and $\frac{105}{71}$, the number of hours required for the whole work, we find that they are the reverse of one another; and this we might have expected, for the quantity of work done in any given time bears an *inverse* ratio to the amount of time in which it is done (76).

Here I have been finding the time of doing the whole work. I give one more Ex., in which it is required to find how much of a given piece of work can be done in any fixed time.

Ex. A cistern is filled by two spouts in 20 and 24 minutes respectively, and emptied by a tap in 30 minutes; what portion of it will be filled in 15 minutes, when they are all left open together, the influx and efflux being uniform?

Since the 1st tap would fill the whole in 20 min.
therefore it would fill $\frac{1}{20}$th in 1 min.
so also, the 2nd „ $\frac{1}{24}$th in 1 min.

therefore $\frac{1}{20} + \frac{1}{24}$ are poured in by both together in 1 minute; but $\frac{1}{30}$ is discharged in 1 minute by the third tap; therefore, subtracting the quantity discharged from that poured in, we have remaining in the cistern at the end of 1 minute, $\frac{1}{20} + \frac{1}{24} - \frac{1}{30}$;

$$\text{and this} = \frac{6+5-4}{120} = \frac{11-4}{120}$$

$$= \frac{7}{120} \text{ in one minute;}$$

therefore in 15 minutes there is $15 \times \frac{7}{120}$ or $\frac{7}{8}$; hence in 15 minutes the *cistern* will be seven-eighths full.

DECIMAL FRACTIONS,

OR

DECIMALS.

91. It is known to all who understand *Numeration*, that where several figures are placed in a horizontal row, so as to form one number, the value of any figure depends upon its distance from the figure nearest to the right, or, as it is commonly called, from the units' place. Thus, if we take the number 6666, we know that the four figures, counting from left to right, have these values respectively—6000, 600, 60, 6; where we observe that the first 6 to the left has 10 times the value of the second—the second, 10 times the value of the third, and so on: or, going from right to left, each figure has one-tenth of the value of the one preceding it; in other words, any figure, when moved from right to left is multiplied by 10 every step, and when moved from left to right is divided by 10 every step. For example, in the number

D C B A
6 6 6 6,

if I move 6 from A to D, or three places to the left, I in reality multiply it by $10 \times 10 \times 10$, or 1000, *i. e.* the 6 becomes 6000. Again, to change a 6 from C to A, or two places to the right, I divide it by 10×10, or 100; *i. e.* the figure which before represented 600 now represents 6.

92. Since, then, it has been shown that successive figures to the right are found by dividing by 10, let this

division be continued beyond the units' place; we ought, therefore, to have, as the value of the first figure to the right of the units' place, $\frac{1}{10}$th of 6, or $\frac{6}{10}$; of the next, $\frac{1}{10}$th of $\frac{6}{10}$, or $\frac{6}{100}$; so, of the next, $\frac{6}{1000}$; of the next, $\frac{6}{10000}$, &c. And when these four additional figures are placed to the right of the units' place, the entire number will be $6666\cdot\overset{\text{EFGH}}{6666}$, where a point has been set after the units' place, to show where the new figures commence. The entire number now consists of 6 thousands, 6 hundreds, 6 tens, 6 units (or 6), 6 tenths, 6 hundredths, 6 thousandths, 6 tenths of thousandths.

Also, observing $\overset{\text{EFGH}}{\cdot 6666}$, it may be seen that the same rule holds that was true of the whole numbers, viz. that to move any figure from right to left is to multiply by 10 every step, and from left to right, is to divide by 10. For example: if I change the second 6 from F to H, or move it *two* places to the right, I change it from $\frac{6}{100}$ to $\frac{6}{10000}$, which $= \frac{6}{100} \div 100$; *i. e.* I have divided it by 10×10, or by 10 *twice.*

Again, if I change the fourth 6 from H to E, or move it *three* places to the left, I change it from $\frac{6}{10000}$ to $\frac{6}{10}$, which $= \frac{6}{10000} \times 1000$; *i. e.* I have multiplied $\overset{\text{H}}{6}$ by $10 \times 10 \times 10$, or by 10 three times; hence the same law holds both on the right and left of the point.

DEF. If a number be multiplied by itself any number of times, it is said to be raised to a power. Thus, the multiplication of $2 \times 2 \times 2$ is otherwise expressed by saying that 2 is raised to the power of 3; and a small figure 3 placed to the right of the 2 and above the line (thus, 2^3) indicates or explains how many factors, 2, have been

multiplied together. In like manner, 10^4 expresses the multiplication of four factors, each 10, and is called the fourth power of 10.

This small figure is called the *Index*, or *Exponent*.

93. It appears that the figures on the right of the point in reality represent fractions, as $\frac{6}{10}$, $\frac{6}{100}$, $\frac{6}{10000}$, &c. all of which have as denominators either 10, or powers of 10; hence they are called *Decimal* Fractions, or Decimals. And the point which is placed to separate the whole numbers from the decimals is called the *decimal point*.

94. We have just seen (91) that in writing down the value which any single figure represents in the whole number 6666, viz. 6000, 600, 60, 6, we place as many ciphers at the right-hand of each 6, as will keep it in the place which it had in the number 6666: so also, in the number ·6666, if we wish to write down the value of each one of these four figures separately, we shall have to place as many ciphers to the left of each 6 as will keep it in the place which it had in the number ·6666, or at its proper distance from the *point*. Hence these four figures, when placed singly, would be

$$\cdot 6, \quad \cdot 06, \quad \cdot 006, \quad \cdot 0006; \qquad \text{(W)}$$

and, as in whole numbers we might go farther to the *left*, and have as the next figure 60000; so, by going farther to the *right* in the decimals, I should have as the next figure ·00006.

Comparing the four quantities in line (W) with the value which we have shown to be due to them, viz. $\frac{6}{10}$, $\frac{6}{100}$, $\frac{6}{1000}$, &c., we have this connection,

$$\frac{6}{10} = \cdot 6 \qquad \frac{6}{100} = \cdot 06 \qquad \frac{6}{1000} = \cdot 006 \qquad \frac{6}{10000} = \cdot 0006 \qquad \text{(X)}$$

where it will be observed, that the number of *ciphers* in the den^{r} of the left-hand side equals the *whole* number of figures after the decimal point, whether ciphers or not, on the right-hand side.

95. Taking this fact as proved when there is but one figure in the num^{r} of the decimal frac^{n}, I shall now show that this is true, whatever be the number of figures in the num^{r} of the frac^{n}; for example, that $\frac{675}{10000} = \cdot 0675$, where there are four *ciphers* in the left-hand den^{r}, and four *figures* after the decimal point.

$$\text{For } \frac{675}{10000} = \frac{600}{10000} + \frac{70}{10000} + \frac{5}{10000}$$
$$= \frac{6}{100} + \frac{7}{1000} + \frac{5}{10000};$$

and these, by what has just been shown, in last Article,

$= \cdot 06 + \cdot 007 + \cdot 0005$
$=$ 6-hundredths + 7-thousandths + 5-tenths of thousandths
$= \cdot 0675;$

i. e. $\frac{675}{10000} = \cdot 0675$, which was the required result.

Hence any frac^{n}, having as a den^{r} any power of 10, can be immediately written as a decimal, by writing down the num^{r} only, and so placing the decimal point that it shall have as many figures on its right-hand, as there are ciphers in the den^{r} of the given frac^{n}. And if the num^{r} does not contain as many figures as there are ciphers in the den^{r},—that is, as many as it is necessary to have after the point, the amount must be made up by placing as many ciphers between the point and the figures taken from the num^{r} as shall complete the desired number.

Thus, $\frac{3275}{1000} = 3 \cdot 275$ $\qquad$ $\frac{743}{100000} = \cdot 00743.$

In the latter Ex. I find it necessary to place two ciphers between the point and the 743, to make the number of figures to the right of the point equal to the number of ciphers in the left-hand denr.

Obs. For the future, in speaking of Decimal Fractions, I shall use the single word *Decimals*; and for Vulgar Fractions, the word *Fractions*.

96. The position of the decimal point determines the value of every figure both on the right and left of it, that is, both of the whole numbers and the decimals. Therefore, to move the point to the right has the same effect as moving all the figures to the left; and to move the point to the left, is equal to moving the figures to the right.

Now, it has been shown (92) that to move a figure one place to the left is to multiply it by 10; therefore if, in any number containing a decimal point, I move the point one place to the right, I in reality multiply every figure in the number, and therefore the entire number, by 10: similarly, if I move the point to the left one place, I divide it by 10. Hence, if I wish to multiply a number containing a decimal point by 10, 10^2, 10^3, &c., I move the point 1, 2, 3, &c. places to the right; and if I wish to divide the number by 10, 10^2, 10^3, &c., I move the point 1, 2, 3, &c. places to the left.

Ex. 3275·468. (Y)

Moving the point two places to the right, the number becomes 327546·8; and it will be found that any figure has now 100 times the value that it had before. The 5 formerly stood for 5 *ones*, or 5, but now for 5 *hundreds*, or 500; the 6 for 6 hundredths, or $\frac{6}{100}$, but now for 6, or $\frac{6}{100} \times 100$; that is, each figure has been multiplied by 100, merely by moving the point two places to the right.

Next, let the point be moved three places to the left, and the number becomes 3·275468; and we then see that the 7, which in (Y) was 70, is now only $\frac{7}{100}$, or $\frac{70}{1000}$; that is, has one-thousandth of its former value: so, also, the 3, which before represented 3000, now represents only 3; hence it appears that the entire number has been divided by 1000, merely by moving the point three places to the left.

CASE I.

97. In the use of Decimals we shall find it necessary to know how to convert decimals into fractions, and fractions into decimals. And it will be seen that there are two classes of fractions, one containing those which can be exactly expressed as decimals, and the other such as cannot. We shall presently show the ground upon which this variety rests; and in this Case shall treat of the first class, which includes all fractions which, when in their lowest terms, have as den^{rs}, numbers containing the figures 2 and 5 as their only factors. These may be called *convertible* fractions.

98. We have seen that a frac^{n} whose den^{r} is a power of 10, can immediately, by inspection, be converted into a decimal. Hence, to bring a fraction into a decimal, I must first, if possible, reduce it to a fraction with such a denominator.

And since fractions can be changed into equivalent fr^{ns} of different den^{rs}, only by multiplying or dividing both num^{r} and den^{r}, my object will be to find some multiplier or divisor which shall transform the den^{r} of any proposed fr^{n} into the required form, 10, 100, 1000, &c.

Any den^r which contains an equal number of both 2's and 5's, will evidently contain as many factors 10, as 2 and 5, and therefore be of the form 10, 100, 1000, &c. Take for example as a den^r $2\times2\times5\times5 = 10\times10 = 100$: and when a fr^n with such den^r is converted into a decimal, there will plainly be as many decimal places as there are 2's or 5's.

Next, let the den^r contain 2's only, or be a power of 2, as $\frac{3}{8}$ or $\frac{3}{2^3}$. Now, if I multiply both num^r and den^r by some power of 10, and divide both num^r and den^r by the original den^r 8, the resulting fr^n will have a den^r of the required form: but if the new num^r can be divided exactly by 8, it follows that the multiplier must contain all the factors of 8, *i. e.* $2\times2\times2$; and since this multiplier contains only 10's, and each 10 contains one 2, therefore I must have three factors 10 to contain the above three 2's; *i. e.* my multiplier must be $10\times10\times10 = 1000$. Using, then, this multiplier, and dividing by the 8, I have the whole process as follows:

$$\frac{3}{8} = \frac{3000}{8000}\left(= \frac{\frac{3000}{8}}{1000}\right) = \frac{375}{1000} = \cdot375 \qquad \text{(A)}$$

and in dividing 3000 by the 8, it is found that the division will not terminate until I have used the 3 ciphers, as was foreseen.

We have just now multiplied num^r and den^r of $\frac{3}{8}$ by 1000 and divided by 8; that is, we have multiplied by $\frac{1000}{8}$, or 125: if, therefore, we at once multiply by 125, we ought to obtain the same result. The work would then be

$$\frac{3}{8} = \frac{3\times125}{8\times125} = \frac{375}{1000} = \cdot375.$$

99. In working Exs. under this Case, a pupil should at first write out the operation as in line (A), omitting the fr^n in

brackets; but in practice the work may be performed mentally, by the method of Short Division.

Since 8 is a divisor, I commence as follows:—8 in 3 units gives no units; I must therefore bring these units into tenths, and say, 8 in 30 tenths, which gives 3 *tenths*, and 6 tenths over; and since 6 tenths = 60 hundredths, 8 in 60 hundredths gives 7 *hundredths*, and 4 hundredths over, or 40 thousandths; 8 in 40 thousandths gives 5 *thousandths;* and there being no remainder, the division terminates: and collecting the three quotients, viz. 3 tenths, 7 hundredths, and 5 thousandths, we plainly have $\frac{3}{8} = \cdot 375$.

In Simple Short Division, as soon as I know where to put the first quotient, I can write down the others in order; so here, as soon as I find the first quotient consists of tenths, I can place the other quotients without further consideration. But in determining the first quotient there is a liability to error; for if a pupil takes the fr^{n} $\frac{3}{80}$, and begins to divide by 80, he may incautiously cut off the ciphers from divisor and dividend, as in Simple Division: but, by placing the fr^{n} thus

$$\begin{array}{r} 80) \;\; 3\cdot 0000 \\ \hline \cdot 0375 \end{array}$$

it will be found that if I cut off a cipher at the end of the dividend, I remove a figure which has no value; whereas, to take 0 from the 80 diminishes it tenfold: I must therefore say, 80 in 3 units gives as quotient 0 units; so also, since 3 units = 30 tenths, 80 in 30 tenths gives 0 tenths. Again, 30 tenths = 300 hundredths, and 80 in 300 hundredths gives 3 hundredths; hence 0 must be put as quotient under the tenths, and 3 under the hundredths—the

remaining quotients will then fall into their proper places by mere Simple Division.

100. If the den^{r} of the fr^{n} be a composite number larger than 12, we may perform the division by breaking the den^{r} into factors, and dividing by them successively. For example

$$\frac{5}{32} = \frac{5}{8\times4} = \frac{1{\cdot}25}{8} \qquad \text{(B)}$$
$$= {\cdot}15625.$$

where the last fraction in (B) was obtained by dividing by 4.

A fraction whose den^{r} has 5's only as its factors, must be reduced to a decimal by a process precisely similar to the one exhibited in (A); and the power of 10, which is to be used as a multiplier of num^{r} and den^{r}, must, by the same reasoning as in (98), have as many factors 10 as there are factors 5 in the den^{r}.

Ex. $\frac{3}{25}$. Here there are two factors 5 in the den^{r}; therefore I shall have to multiply by 100, and divide by 25, or multiplying by $\frac{100}{25}$, that is, by 4, I write

$$\frac{3}{25} = \frac{12}{100} = {\cdot}12 \qquad \text{or } \frac{3}{25} = \left(\frac{3}{5\times5}\right) = \frac{{\cdot}6}{5} = {\cdot}12$$

101. Reverting to the fr^{n} $\frac{3}{80}$, or $\frac{3}{2\times2\times2\times2\times5}$, we may see, that since four 2's have to be removed from the den^{r}, the power of 10, which is here to be the multiplier, must contain four factors 10, and the single 5 will also be cancelled out by one of the factors 5 in this multiplier. Hence we see that in this class of fractions, which we have called *convertible*, the multiplier required to convert such a fr^{n} into a decimal must have as many factors 10 as the larger number of 2's or 5's in the den^{r}.

102. Coming now to what we may call the *inconvertible* fractions, *i. e.* those which, when reduced to lowest terms, contain other factors than 2 and 5 in the den^{r}, we see at once, that since a power of 10 is the only multiplier which can render the den^{r} of the required form; therefore any factors in the den^{r}, as 3, 7, 9, &c., which are not contained in 10 cannot be cancelled out, as the 2's and 5's were in (98): hence the division, such as in (A), will not terminate. But since the successive quotients diminish tenfold in value every step, so that the eighth to the right of the point represents hundredths of millionths, we generally pursue the division till we obtain 7 figures in the decimal, and then consider the remaining quotients too small in value to be much appreciated.

Moreover, whatever be the factor which cannot be cancelled out of the den^{r}, we know that in dividing by any divisor, the remainder must always be at least one less than the divisor: therefore if any divisor, as 7, be the factor which causes the division not to terminate, there can be but six remainders; and hence, when 7 divisions have been performed, one of these six remainders must come over again, or recur: and if the remainder, which, of course, influences the dividend, recurs, the quotients will also recur, and we shall then have the same set of figures *recurring*, as it is termed.

Ex. To convert $\frac{6}{7}$ into a decimal.

Working mentally, as in (99), I have

$$\frac{6}{7} = \cdot 857142857142, \text{ &c.}$$

where it appears that the figures 857142 will recur, to whatever length the division may be carried: hence such decimals are called *Recurring Decimals*, and the set of quotients which recurs is called a *Period*. If the period

consists of but one figure, that single figure is written with a dot (·) over it in place of the whole decimal; but if there be more figures than one, the period is written with a dot over the first and last of its figures.

Thus, $\frac{6}{7} = \cdot\dot{8}57142\dot{8}$

$\frac{1}{3} = \cdot333\ldots\ldots$ or $= \cdot\dot{3}$

$\frac{1}{9} = \cdot1111\ldots\ldots$ or $\cdot\dot{1}$

$\frac{3}{22} = \cdot13636\ldots\ldots$

or $\cdot1\dot{3}\dot{6}$

Obs. The places which are occupied by the figures to the right of the decimal point are often called *decimal places*. Thus, in the Ex. $\frac{6}{7}$, we say that the division was carried to six places before the figures began to recur.—(See Appendix, Art. Circulating Decimals).

The following equalities are worth remembering.

$\frac{1}{8} = \cdot125$ $\quad$ $\frac{1}{4} = \cdot25$

$\frac{3}{8} = \cdot375$

$\frac{1}{2} = \cdot5$

$\frac{5}{8} = \cdot625$

$\frac{7}{8} = \cdot875$ $\quad$ $\frac{3}{4} = \cdot75$

Their utility appears as follows:—if in converting a fraction to a decimal I have a divisor 8, and I come to the last figure in the dividend, one of the above equalities will enable me to write down the remaining quotients immediately.

Thus, if I have to reduce $\frac{75}{8}$ to a decimal. I find that after one quotient 9 has been obtained, 3 is my last dividend; hence, since $3 \div 8 = \cdot375$, I can write $\frac{75}{8} = 9\cdot375$, without *performing* the division for the last three quotients. Similarly, $\frac{11\cdot7}{8} = 1\cdot4625$, where the last three figures were obtained, as before, by remembering the value of $5 \div 8$. So also, $\frac{75}{4} = 18\cdot75$,—the last two quotients being written without dividing.

CASE II.

TO CONVERT TERMINATING DECIMALS INTO VULGAR FRACTIONS.

103. It will be easily seen that a number expressed as a decimal can be immediately exhibited as a vulgar fraction. For, since it has been shown (95) that $\frac{675}{10000} = \cdot0675$, therefore we can reverse the process, and say that $\cdot0675$, when converted into a fr^{n}, becomes $\frac{675}{10000}$; and the correctness of this conversion may be proved thus:

$$\cdot0675 = \frac{6}{100} + \frac{7}{1000} + \frac{5}{10000}$$

$$= \text{(when reduced to L. C. D.)} \frac{600 + 70 + 5}{10000}$$

$$= \frac{675}{10000}. \qquad \text{(C)}$$

$$\text{So also, } 3\cdot0275 = 3 + \frac{0}{10} + \frac{2}{100} + \frac{7}{1000} + \frac{5}{10000}$$

$$= \frac{30000 + 200 + 70 + 5}{10000} = \frac{30275}{10000}, \qquad \text{(D)}$$

a fraction greater than 1, as might have been foreseen, because $3\cdot0275$ is partly a whole number and partly a decimal.

DEF. In a whole or mixed number, the part which is not fractional is called *Integral*, or an *Integer*.

104. In (C) we observe that the num^{r} of the vulgar fr^{n} into which we have changed the decimal, before it is reduced to lowest terms, consists of the figures in the given decimal, excluding ciphers; and the den^{r} is 1, followed by as many ciphers as there are decimal places.

But in (D), where a cipher occurs in the given decimal $3\cdot0275$, it remains in the num^{r} of the equivalent improper fr^{n}; and the den^{r} is formed as before.

The above fractions, of course, may be reduced to lowest terms; but I have left them in their present shape in order to show the connection between the given decimal and the fractional form into which it could be converted by inspection. Reducing them, we have

$$25)\ \frac{675}{10000} = \frac{27}{400};\ \text{and}\ \frac{30275}{10000} = 3\frac{275}{10000} = 3\frac{11}{400}.$$

If the given number be partly a whole number and partly a decimal, we may leave the integral part unaltered, and then the resulting fr^{n} will appear at once as a mixed number. Thus, $76{\cdot}0725 = 76\frac{725}{10000} = 76\frac{149}{2000}$.

The Exs. in Case III. have been included in Case I.

CASE IV.

TO CONVERT RECURRING DECIMALS INTO VULGAR FRACTIONS.

105. Though at first it may be thought that a *non-terminating* decimal cannot be accurately represented by a fr^{n}; yet, since every recurring decimal is formed from one of that class of fractions which was discussed in Case II., therefore every such decimal can of course be made to resume the shape from which it was derived: and the accuracy which was lost in the change from a fraction to a decimal is restored by this reconversion to the original fraction.

106. Circulating Decimals are of two kinds; one in which the whole of the figures repeat, as

$$\cdot 363636\ldots\ldots \text{or } \cdot\dot{3}\dot{6};$$

and the other, in which some of the figures to the right of the point are not repeated. These figures, of course, always

stand to the left of the circulating part, because when a decimal once begins to circulate, it continues to do so. An Ex. of this kind is

$$\cdot 754365365\ldots\ldots \text{ or } \cdot 754\dot{3}6\dot{5}.$$

The former kind is called a *Pure*, and the latter a *Mixed* Circulating Decimal.

107. The best method of working Exs. under this Case will not be intelligible without an acquaintance with one or two facts in algebra. A letter of the alphabet may be used to represent a quantity, the value of which we have to find, as in (73). Thus, I may let F stand for or represent the fraction which is equivalent to any circulating decimal; and it is for me to ascertain the value of this F in any particular Example.

Also, a number placed close to this F, and with no sign connecting F and the number, is a multiplier of the F: thus, 10F, $\frac{2}{3}$F, mean 10 times F, $\frac{2}{3}$ times F, or $\frac{2}{3}$ of F; and they have the same value as though they were written $10 \times F$, $\frac{2}{3} \times F$. F repeated once is not written 1F, as we should write 1s., but only F. These multipliers 10, $\frac{2}{3}$, and 1, are called *coefficients*. Moreover, since in the quantity 10F, F is the den^{n}, and the 10 tells us how many times F is taken, therefore just as the 7 in 7s. tells us how many shillings are taken, so I can add to this 10F, or subtract from it, any number of quantities of the same den^{n}, *i. e.* any number of times F.

Thus, $10F + 3F = 13F$ $\qquad$ $10F - 3F = 7F$
and $10F + F = 11F$ $\qquad$ $10F - F = 9F$.

108. I now proceed to find the fr^{n} which is equivalent to any circulating decimal, as

Ex. I. $\cdot\dot{3}\dot{6}$ or $\cdot 363636\ldots\ldots$

Let $F = \cdot 363636\ldots\ldots$ (E)

Now multiply both sides of this equation by 100, and the equality will not be disturbed. The left-hand side becomes $100 \times F$, or 100F: the value of the right-hand side is found (90) by moving the point two places to the right, and the decimal then becomes 36·3636.....

$$\text{therefore, } 100F = 36{\cdot}3636..... \quad (F)$$
$$\text{also, } F = \quad {\cdot}363636..... \quad (E)$$

therefore, subtracting (E) from (F) we have

$$100F - F = 36 + {\cdot}3636..... - {\cdot}363636..... \quad (G).$$

Now, as in (46) we found the difference, first of the fractional parts, and then of the whole numbers, so here we find that the difference of the fractional parts is 0; for, since both decimal parts commence alike and go on for ever, their difference is 0; therefore line (G) becomes

$$100F - F = 36 + 0$$
$$\text{or, } 99F = 36$$

and transferring 99 into the right-hand den^{r}, we have

$$F = \frac{36}{99} \qquad (H)$$
$$= \frac{4}{11}$$

By observing (H) we learn that the fraction which represents the value of the circulating decimal $\cdot\dot{3}\dot{6}$ has for its num^{r} the figures in the period, viz. 36, and for the den^{r}, as many 9's as there are figures in that period.

In accordance with this rule we may therefore, by inspection, find the value of any circulating decimal which contains only such figures as do repeat.

$$\text{Thus, } .\dot{6} = \frac{6}{9} = \frac{2}{3}$$
$$\text{and } \cdot\dot{1}4\dot{7} = \frac{147}{999} = \frac{49}{333}.$$

109. Again, noticing (E) and (F) we see that the figures to the right of the decimal point are the same in both lines; and this result was produced by so multiplying both sides of the equation (E), that the point might be made to pass over one period; in this case it had to be moved two places to the right; that is, I had to multiply by 10×10, or 100. If the period had contained four figures, I should have multiplied by 10^4, or 10000. I will take as an Example

To express $\cdot\dot{2}21\dot{6}$ as a fraction.

Let $F = \cdot 22162216 \ldots\ldots$ (I)

therefore, $10000F = 2216 \cdot 2216 \ldots\ldots$ (K)

Subtracting (I) from (K), we have

$$9999F = 2216;$$

$$\text{therefore, } F = \frac{2216}{9999}:$$

and it is plain that this result might have been written down at once by taking as numr of the required frn the period 2216, and as denr a number consisting of as many figures 9 as there are figures in this period.

110. We now proceed to convert a mixed circulating decimal into a fraction. The nature of the process is the same as in the last article; but one step more is required, the reason of which will appear in the operation.

To convert $\cdot 32\dot{7}1\dot{5}$ into a fraction.

Let $F = \cdot 32715715 \ldots\ldots$ (L)

Then, as before (109), carrying the decimal point over the period, or five places to the right—*i.e.* multiplying both sides by 100000, I have

$$100000F = 32715 \cdot 715 \ldots\ldots \text{ (M)}$$

It now appears that I cannot subtract (L) from (M), so as to get rid of the decimal parts, because the quantities to

the right of the point are not the same: but I see that I can very readily obtain another equation which shall have the decimal part the same as in (M); for if in (L) I multiply both sides by 100, *i. e.* move the point two places to the right, viz. over the non-recurring part alone, I shall have

$$100F = 32{\cdot}715715\ldots\ldots \quad \text{(N)}$$

therefore, subtracting (N) from (M), I have

$$100000F - 100F = 32715{\cdot}715\ldots\ldots - 32{\cdot}715715\ldots\ldots \quad \text{(O)}$$

$$\text{or } 99900F = 32715 - 32;$$

$$\text{therefore, } F = \frac{32715 - 32}{99900} \quad \text{(P)}$$

$$= \frac{32683}{99900}.$$

111. In the line (O) I learn that the larger coefficient of F has as many ciphers as there are figures from the point to the end of the first period; and the smaller coefficient has 2 ciphers, *i. e.* as many as there are figures in the non-recurring part: also, when the subtraction is *performed* in the next line, the coefficient of F has as many ciphers as there were in this said smaller coefficient, *i. e.* as many as there are figures in the non-recurring part: and the remaining figures to the left are all nines, and as many in number as there are figures in the period. Hence, observing the right-hand side of (P), I learn that in converting a mixed circulating decimal into a fraction, I obtain as num^{r}, "the figures of the given decimal to the end of the first period — the figures in the non-recurring part;" and as den^{r}, as many figures 9 as there are figures in the period, followed by as many ciphers as there are figures in the non-recurring part. If there be an integer in the given recurring decimal, I may omit it while finding the value of the recurring part; and afterwards by inserting it I shall have

the resulting fraction a mixed number; or I may retain it throughout, as in Ex. II., and the result will be an improper fraction.

I will write down one more Ex. of each kind of circulating decimals, in the form in which they ought to be worked.

Ex. I. Find the fraction equivalent to $\cdot\dot{6}3\dot{9}$.

$$\text{Let } F = \cdot 639639639\ldots\ldots\ldots \text{ (I.)}$$

multiplying by 1000,

$$1000F = 639\cdot 639639\ldots\ldots\ldots \text{ (II.)}$$

Subtracting I.) from (II.),

$$999F = 639,$$

$$\text{or } F = \frac{639}{999} = \frac{213}{333} = \frac{71}{111}.$$

Ex. II. Find the fraction equivalent to $2\cdot 0\dot{3}4\dot{5}$.

$$\text{Let } F = 2.0345345\ldots\ldots\ldots \text{ (III.)}$$

multiplying by 10000,

$$10000F = 20345\cdot 345\ldots\ldots\ldots \text{ (IV.)}$$

Again, multiplying (III.) by 10, $10F = 20\cdot 345345\ldots\ldots\ldots$ (V.)

and subtracting (V.) from (IV.),

$$9990F = 20345 - 20$$

$$\text{therefore, } F = \frac{20325}{9990} = \frac{6775}{3330} = \frac{1355}{666}$$

$$= 2\tfrac{3\,3\,3}{6\,6\,6};$$

or omitting the integral part, as described above, I have

$$F = \frac{345}{9990} = \frac{69}{1998} = \frac{23}{666},$$

therefore, the entire quantity $= 2\tfrac{3\,3\,3}{6\,6\,6}$, as before.

CASE V.

ADDITION.

112. Since we already know that quantities cannot be added together unless they be of the same den^{n}, so, in the addition of decimals, we must take care to add tenths to tenths, hundredths to hundredths, and so on. And this

will be readily done, if we place all the numbers under one another, so that the decimal points are in a vertical row. By observing the Ex. worked below, we notice that the tenths are all placed under one another, as are also the hundredths, thousandths, &c.

Ex. I.
$$\begin{array}{r} 732{\cdot}416 \\ {\cdot}084 \\ {\cdot}000007 \\ 93{\cdot}268 \\ 2708{\cdot}4153 \\ \hline 3534{\cdot}183307 \end{array}$$

Adding up the third row to the right of the point, which consists of thousandths, I find that it amounts to 23 thousandths, or $\frac{23}{1000}$, which $= \frac{20}{1000} + \frac{3}{1000} = \frac{2}{100} + \frac{3}{1000}$; I therefore put down the 3 thousandths, and carry the 2 hundredths to the next column, which consists of hundredths; and since this step is just such as would be performed in Simple Addition, it is plain that all the rest of the work may be performed by that Rule.

113. But if in the quantities to be added there are circulating decimals, we may either, 1st, convert into a fr^{n} each circulating decimal, and having found the sum of all these fractions, reduce the result to a circulating decimal: or, 2ndly, (and this method is the better,) write down the recurring decimals at length, to as many places as will include twice the longest period; observe where there are two vertical columns alike, though not necessarily close to each other; commence the addition three or four places to the right of the second of these two similar columns, and complete the addition, as before: the sum found will be seen to be carried far enough to enable a pupil to detect the period in the answer.

Ex. II. Find the sum of $\cdot\dot{7}14285 + \cdot\dot{9}2857\dot{1}4 + 20{\cdot}0\dot{9}2\dot{5} + 5{\cdot}4\dot{0}4761\dot{9}$.

Writing these at length, and working according to the above directions, I have

$$\begin{array}{rl} & \begin{array}{cc} \text{A} & \text{B} \end{array} \\ & \cdot 714285714285\ldots. \\ & \cdot 9285714285714\ldots. \\ 20 & \cdot 0925925925925\ldots. \\ 5 & \cdot 4047619047619\ldots. \\ \hline 27 & \cdot 14021164020 \end{array}$$

Here I observe that the first two similar vertical columns are the second and the eighth, marked (A) and (B). Commencing the addition at the third column beyond (B), I find that I have figures enough in the result to show that the sum of the given circulating decimals is $27\cdot1\dot{4}02116\dot{}$.

CASE VI.

SUBTRACTION.

114. As Addition of Decimals was shown to be only an extension of Simple Addition, so Subtraction of Decimals is of the same nature with Simple Subtraction; but we must take care how we subtract, when the lower line contains more decimal places than the upper.

Ex. Find the value of $18\cdot0426 - 2\cdot005417$.

Placing the two quantities one under the other, so that the points are in a vertical row, we shall, as in Addition, have tenths under tenths, &c.

As there are no figures in the 5th and 6th places in the upper line, I may place ciphers there, and since there are no figures to the right of these ciphers, the value of the decimal will plainly remain unaltered.

A	B
18·042600	600
2·005417	417
16·036183	183

If we now take the last three figures in each row as whole numbers, so as to form a Simple Subtraction Sum, we find the difference to be 183: so also, since the 600 and 417 in (A) both represent millionths, their difference is 183 millionths: similarly the 42 and 5 are both thousandths, and their difference = 37 thousandths: hence we see that if the subtraction be performed in (A) as in a common Ex. in Subtraction, the result will be correct.

115. If an Ex. in Subtraction contain circulating decimals, we must, as in Addition, write down the periods sufficient to contain twice the longest period, and commence the subtraction about five places beyond the latter one of the two similar rows; the remainder which will recur will be found to contain figures enough to enable us to detect the *period.*

Ex. Find the value of $24\cdot\dot{4}0\dot{7} - 5\cdot9\dot{1}0769\dot{7}\dot{3}$.

Here, writing five of the periods in the upper line, and two in the lower line, we have

$$\begin{array}{l} \overset{A}{24\cdot407407407407407} \overset{B}{\ldots\ldots} \\ 5\cdot98076973076973 \ldots\ldots \\ \hline 18\cdot4266376766377 \end{array}$$

The columns marked A and B are the first pair which are similar; I therefore commence the subtraction at the fifth row beyond B: and the remainder, which contains the circulating decimal, is evidently $18\cdot42\dot{6}63767\dot{3}$.

By observing the Ex. in Addition of circulating decimals, I find that the figures which should form the second period are somewhat below the value of those in the first period; but by commencing with the addition a little more to the right, I should have figures to *carry*, as it is called, which would have made the second period correct as well as the first. So in the Ex. just worked in Subtraction, the figures which should form the second period are too large; but by commencing the subtraction a few more places to the right, the process of *borrowing* would make the figures of the second period quite correct.

CASE VII.

MULTIPLICATION.

116. The rule for the multiplication of decimals is to be found by multiplying the vulgar fractions equivalent to any given decimals, and then observing the nature of the product so obtained.

Ex. To find the product of $3\cdot275$ and $18\cdot03$.

We have $$3\cdot275\times18\cdot03 = \frac{3275}{1000}\times\frac{1803}{100}$$

$$= \frac{3275\times1803}{100000} = \frac{5904825}{100000} \quad (Q)$$

From (Q) we learn that the product of the given decimals is a decimal fraction which has for its numerator the product of the given numbers treated as integers, and for its denominator 1, followed by as many ciphers

as there are decimal places in *both* the multiplier and multiplicand; and such a fraction is of course immediately (95) convertible into a decimal, which shall have as many decimal places as this fraction has ciphers. Hence, as this Ex. differs from no other which contains only terminating decimals, we conclude that the product of two such decimals is found by multiplying them together as in whole numbers, and pointing off in the product as many places as are found in both multiplier and multiplicand.

117. Also, if it be required to multiply together more than two numbers involving decimals, the same rule must be observed. For suppose that four numbers were given, whose product was required: the product of the first pair might be found, as above; then this product and the third number might be multiplied; and, lastly, this second product and the fourth number: and so on for any number of quantities; hence we may find the product of any number of decimal quantities, as in the previous Example.

Ex. II. Find the value of $\cdot0095 \times 2\cdot07 \times 7\cdot06 \times \cdot0081$.

$$\text{The product} = \frac{95}{10000} \times \frac{207}{100} \times \frac{706}{100} \times \frac{81}{10000}$$

$$= \frac{95 \times 207 \times 706 \times 81}{1000\,000\,000\,000} = \frac{1124562690}{1000\,000\,000\,000}$$

$$= \cdot001124562690$$

where the number of decimal places, viz. 12 = sum of the numbers of places in the four given quantities.

118. Next, let the decimals to be multiplied together be either one or both of them circulating. These may be converted into equivalent fractions; and then, after having multiplied them, we may convert the product into a circulating decimal; or the product of the decimals themselves may be found by Simple Multiplication, provided that there be taken a sufficient number of figures to enable us to ascertain the period in the product. We will work two Exs. to illustrate both methods.

Ex. III. Find the value of $\cdot\dot{3}\dot{6}\times 75$.

By the first method I have

$$\cdot\dot{3}\dot{6}\times 75 = \frac{\overset{4}{\cancel{36}}}{\underset{11}{\cancel{99}}}\times 75 = \frac{300}{11} = 27\cdot2727\ldots\ldots$$

By the second method, I write down the period three times, so that there may be figures enough in the product to show its period.

$$\begin{array}{r} \cdot363636\ldots. \\ 75 \\ \hline 1818180 \\ 2545452 \\ \hline 27\cdot272700 \end{array}$$

If a fourth period had been written in the multiplicand, the figures carried from the multiplication of it would have caused the period 27 to have been seen in the fifth and sixth decimal places in the product: and every additional period in the multiplicand would have produced one more period in the product: hence, since the number of periods, 36, in the multiplicand is unlimited, so also will be the number of periods, 27, in the product; *i. e.* the product is $27\cdot\dot{2}\dot{7}$.

Ex. IV. Find the value of $37\cdot\dot{3}\times 9\cdot1\dot{6}$.

Upon working this Ex. by ordinary Multiplication, I find that even if I write down five periods in the multiplicand, and four in the multiplier, yet the product is such that the learner would hardly detect the period in it; and if the periods had contained several figures, the work would be exceedingly heavy; and since in such Exs. involving two or more recurring decimals, the former method is the better, I give it alone. I then have

$$\begin{aligned} 37\cdot\dot{3}\times 9\cdot1\dot{6} &= 37\tfrac{3}{9}\times 9\tfrac{1}{6} \qquad \text{by (111)} \\ &= 37\tfrac{1}{3}\times 9\tfrac{1}{6} \\ &= \frac{\overset{56}{\cancel{112}}}{3}\times\frac{55}{\underset{3}{\cancel{6}}} \\ &= \frac{3080}{9} = 342\cdot\dot{2} \end{aligned}$$

CASE VIII.

DIVISION.

119. In the Exs. under this Case, either dividend or divisor, or both, may be a decimal. I will give one Ex. of each variety.

Ex. I. Find the value of $37{\cdot}5 \div 84$.

Where, as in this Ex., the divisor is a composite number containing no prime factor greater than 12, we can divide by successive factors, as in (100); thus:

$$\frac{37{\cdot}5}{84}\left(=\frac{37{\cdot}5}{12\times 7}\right)=\frac{3{\cdot}125}{7}=\cdot 44642857142, \text{ \&c.}$$
$$=\cdot 4464\dot{2}857\dot{1}$$

120. Also, since we have shown that by moving the point in any decimal from left to right, we in reality multiply that decimal by a power of 10; therefore if we have a decimal quantity as divisor, we can write the sum as a fraction $\left(\frac{\text{dividend}}{\text{divisor}}\right)$ and remove the point so many places to the right in both divisor and dividend as will cause the divisor to be a whole number. By this process we shall merely have multiplied both num^{r} and den^{r} of a fr^{n} by the same power of 10. The remaining part of the division will be merely as in an Ex. under Case I.; and the position of the point will be determined as in that Case.

Ex. II. Find the value of $716{\cdot}343069 \div 27{\cdot}69$.

$$\text{The quotient} = \frac{716{\cdot}343069}{27{\cdot}69}=\frac{71634{\cdot}3069}{2769}=25{\cdot}8701 \qquad (R)$$

and the correctness of this position of the point may be shown immediately: for throwing the 27·69 in the first fraction into the numerator of the right-hand side, I have

$$716{\cdot}343069 = 25{\cdot}8701 \times 27{\cdot}69;$$

where I observe, that if I had to form the product expressed in the right side of the equation, I should have 6 decimal places in it, as I see to be the case in the left-hand side: hence it is clear that the point was rightly placed in the quotient 25·8701. And further, observing the left-hand fraction in (R), I see that by subtracting the number of decimal places in the divisor from that in the dividend, I obtain the number of places in the quotient. Hence we take it as a Rule, that the division should be performed as in whole numbers, and that there should be pointed off in the quotient as many decimal places, as the number in the dividend exceeds the number in the divisor.

121. If, however, the number of places in the dividend be less than that in the divisor, there must be appended as many ciphers to the dividend as shall make up the number of places in the dividend at least equal to the number of places in the divisor. If the dividend be a whole number, a point must be placed after the units' place, and then ciphers may be appended: and it is clear that these ciphers will not in any way alter the value of the dividend.

Ex. III. $971{\cdot}7 \div {\cdot}123$.

Here appending ciphers and working as in Simple Long Division, I have

$$\begin{array}{rll} {\cdot}123) & 971.700 & (7900 \\ & 861 & \\ & \overline{1107} & \\ & 1107 & \\ & \overline{00} & \end{array}$$

and since the number of decimal places in divisor and dividend is the same, therefore the number of places in the quotient will be 0; or the quotient will be an integer.

Ex. IV. $62{\cdot}5 \div {\cdot}025$.

This Ex. I work in two ways; first, by removing the point three places to the right in numerator and denominator, as in (96); and secondly, by converting the given decimals into fractions, and performing the division by the usual method.

$$\frac{62{\cdot}5}{{\cdot}025} = \frac{62500}{25} = \frac{12500}{5}$$

$$= 2500$$

$$\text{or } \frac{62{\cdot}5}{{\cdot}025} = \frac{625}{10} \div \frac{25}{1000} = \frac{625}{10} \times \frac{\overset{100}{1000}}{25}$$

$$= \frac{62500}{25}$$

$$= 2500, \text{ as before.}$$

122. If either or both of the given decimals circulate, the circulator may be converted into a proper or improper fr^{n},

as the case may be, and the division proceeded with as in fractions; or the circulator may be left unaltered, if in the dividend, and only the divisor be converted into a fr^{n}, and the division then performed.

Ex. I. Find the value of $75 \div \cdot\dot{1}4\dot{8}$.

By (108) $\cdot\dot{1}4\dot{8} = \frac{148}{999} = \frac{4}{27}$ (in lowest terms)

therefore $\frac{75}{\cdot\dot{1}4\dot{8}} = \frac{75}{\frac{4}{27}} = \frac{75 \times 27}{4} = \frac{2025}{4} = 506{\cdot}25.$

Ex. II. Find the quotient of $\cdot 96\dot{3}4\dot{5}$ when divided by $\dot{3}$.

$$\cdot\dot{3} = \frac{3}{9} = \frac{1}{3}$$

therefore $\frac{\cdot 96\dot{3}4\dot{5}}{\cdot\dot{3}} = \frac{\cdot 96\dot{3}4\dot{5}}{\frac{1}{3}} = (\cdot 96345345)\ldots\ldots \times 3$

$= 2{\cdot}89036035\ldots\ldots$

$= 2{\cdot}890\dot{3}6.$

It has been stated in (114) and (121) that ciphers may be written after the last figure of a decimal without altering its value: similarly, they may be cut off without affecting it; and this is generally done, if the decimal resulting in any Ex. have ciphers at the end. Thus, in Art. 117, Ex. II., I might have removed the cipher which is at the end of the final decimal; only that a pupil would have thought that there were pointed off only 11 decimal places, instead of 12: but, by referring to the last vulgar fraction used in that Ex., I find that it might have been reduced to lower terms, by dividing numerator and denominator by 10; and there would then have been but 11 ciphers in the den^{r}, and consequently 11 places in the decimal which follows.

CASE IX.

123. This Case corresponds to Case XII. in Vulgar Fractions; and merely requires that the decimal quantity should be reduced to successive lower denominations, until we have either no decimal part remaining, or until we reach the lowest denomination used.

Ex. I. Express in positive terms ·375 of £1.

·375 of £1 = ·375 × 20s. = 7·500 sh.;
and ·5 sh. = ·5 × 12 d. = 6·0 d. = 6 d.;
therefore ·375 of £1 = 7s. 6d.

£
·375
20
7·500 sh.
12
6·0 d.

The work may be written out thus: where it is plain that at the end of each successive multiplication I point off as many decimal places as there were in the preceding line, because there are no decimal places in the multiplier.

It will be seen that in both the above operations I have omitted the two ciphers at the end of the decimal part in the shillings, for the reason given at the close of the last article.

By using the equalities mentioned at the end of (102), and referring to the aliquot parts of different denominations given in (64), we might have worked this Ex. very briefly.

Thus, ·375 of £1 = $\frac{3}{8}$ of £1 = 7s. 6d.

Ex. II. Find the value of 7·14685 of 5s. 6¾d.

Here, since the concrete number is expressed in several denominations, I must either reduce the decimal to a fraction, or reduce the 5s. 6¾d. to the fraction of a penny, and then perform the multiplication.

By the second method:

7·14685 × 5s. 6¾d. = 7·14685 × 66¾d.
= 7·14685 × 66·75d.
= 477·0522375d.
= 39s. 9·0522375d.
Answer. = £1 19s. 9·0522375d.

Just as in (63) I left the remainder as a fractional part of a penny, so here I leave it as a decimal fraction of a penny.

124. If any circulating decimals occur in Exs. under this Case, we must reduce them to fractions, and proceed as before.

Ex. III. Find the value of $\cdot\dot{5}$ of £1 11s. 4d.

$$\cdot\dot{5} \text{ of £1 11s. 4d.} = \frac{5}{9} \text{ of } 31\tfrac{1}{3}\text{sh.}$$

$$= \frac{5}{9} \times \frac{94}{3}\text{ sh.} = \frac{470}{9\times 3}\text{ sh.} = \frac{156{\cdot}666\ldots.}{9}\text{ sh.}$$

$$= 17{\cdot}4074074\ldots.\text{sh.}$$

and $\cdot 407407\ldots.\text{sh.} = (\cdot 407407\ldots.) \times 12\text{d.}$

$$= 4{\cdot}888884\ldots.\text{d.}$$

$$= \cdot 4\cdot\dot{8}\text{d.}$$

$$= 4\tfrac{8}{9}\text{d.}$$

therefore $\cdot\dot{5}$ of £1 11s. 4d. = 17s. $4\tfrac{8}{9}$d.

CASE X.

125. This Case is similar to Case XIII. in Vulgar Fractions.

Ex. I. Reduce 4s. $6\tfrac{1}{4}$d. to the decimal of a sovereign.

In working this Ex. we shall first reduce the former of the given quantities to the fraction of the latter, and then convert into a decimal the vulgar fraction connecting the two quantities.

$$\frac{4\text{s. }6\tfrac{1}{4}\text{d.}}{1\text{£}} = \frac{54\tfrac{1}{4}\text{d.}}{240\text{d.}} = \frac{109}{480} = \frac{13{\cdot}625}{60}$$

$$= \frac{2{\cdot}270833\ldots.}{10}$$

$$= \cdot 227083\dot{3}\,;$$

or 4s. $6\tfrac{1}{4}$d. = $\cdot 227083\dot{3}$ of £1.

MISCELLANEOUS EXAMPLES.

126. The same remark applies to these Miscellaneous Examples in Decimals that applied to the corresponding Exs. in Fractions: and the only general assistance that can be given to a pupil is to show him the neatest method of performing the operations required.

Ex. I. B 3. Multiply £8 17s. 6d. by 75·25, and reduce the result to the decimal of £100.

$$\text{Here } \frac{(£8\ 17s.\ 6d.) \times (75{\cdot}25)}{£100} = \frac{£8\frac{7}{8} \times 75{\cdot}25}{£100}$$

$$= \frac{8{\cdot}875 \times 75{\cdot}25}{100}$$

$$= \frac{667{\cdot}84375}{100}$$

$$= 6{\cdot}6784375$$

Ex. II. C 1. Reduce $\frac{3{\cdot}275}{405}$ of $\frac{2{\cdot}5}{{\cdot}075} \times \frac{3{\cdot}125}{11} \times \frac{9}{9{\cdot}375}$ to a simple quantity.

Moving the point three places to the right in two numerators and in two denominators, and again one place to the right in the numerator and the denominator of the second fraction, the expression becomes

$$\frac{\overset{131}{\cancel{655}}}{\underset{81}{\cancel{405}}} \times \frac{\overset{1}{\cancel{25}}}{\underset{6}{\cancel{30}}} \times \frac{\overset{1}{\cancel{3125}}}{11} \times \frac{\overset{1}{\cancel{9}}}{\underset{3}{\cancel{9375}}} = \frac{131}{9 \times 6 \times 11 \times 3}$$

$$= \frac{11{\cdot}90909....}{9 \times 6 \times 3}$$

$$= \frac{1{\cdot}32323232....}{6 \times 3}$$

$$= \frac{{\cdot}22053872053872....}{3}$$

$$= {\cdot}07351290684624, \text{ \&c.}$$

Since a very large number of decimals is non-terminating, it might seem that vulgar fractions, which are always expressed in finite terms, would be preferable for every

purpose. But this is not the case, for decimals have one advantage over fractions from the following consideration.

In ascertaining the comparative value of two or more fractional quantities, if they be expressed as vulgar fractions, it is necessary to reduce them to a C. D.; but if they are represented as decimals, mere inspection will detect their comparative value, as readily as can be done in whole numbers.

For example, if we have to compare $\frac{7}{15}$, $\frac{11}{25}$, $\frac{13}{28}$, we cannot see which is the largest, and which the least, without reducing them to a C. D.: but if the quantities had been written in their decimal form, viz.

·4666.....; ·44; ·4642857i,

we could see *at once* that the first is the largest; the last one is the next; and the middle one is the smallest: therefore, in order of magnitude they are $\frac{7}{15}$, $\frac{13}{28}$, $\frac{11}{25}$. The decimal form is more especially useful, when several fractional quantities are arranged in a table, and where it is requisite to be able to compare the different quantities at a glance.

We may take as an Ex. the accompanying table, in which the numbers represent the comparative weights of equal bulks of different substances.

Sheet Glass	3·33
Plate Glass	2·5
Marble	2·716
Quartz	2·6
Rock Salt.....	1·92
Ivory	1·917
Ice at 0°	·926
Water at 60°	1·

Though the 5th and 6th numbers are very nearly equal, yet it can be seen at once that the 5th is larger than the 6th by three thousandths: as fractions, these quantities would have been written, $1\frac{13}{25}$; and $1\frac{810}{1000}$; and the difference could not be ascertained by inspection.

PRACTICE.

127. PRACTICE is a Rule which endeavours to show the readiest method of finding the cost of any number of articles at a certain price: and the work exhibited in any Ex. consists of a series of amounts such as a person would try to obtain, if he were working the question mentally.

Thus, if I had to find the value of 56 lbs. at 1½d. each, I should say, 54 at 1d. = 54d. = 4s. 6d.; and 54 at ½d. = 54 halfpence = 27d. = 2s. 3d.: and therefore 54 at 1½d. = 4s. 6d. + 2s. 3d. = 6s. 9d.

128. But if the number of articles had been much larger, or the price much greater, the value of them could not readily have been obtained mentally: we therefore, in Practice, use the above method of mental calculation, but we write down the successive results, and find their sum for the final result.

The Exs. to be worked under this Rule may be arranged as follows:—

When the price is under a shilling, as

Ex. I. 4108 at 7¼d.

When the price is between 1s. and £1, as

Ex. II. 4103 at 7s. 5½d.

Ex. III. 6009 at 19s. 5½d.

When the price consists of more than £1, as

Ex. IV. 7111 at £1 17s. 4½d.

Ex. V. 4013 at £12 7s. 0½d.

When there is a fraction in the given number of quantities, as

Ex. VI. $6583\frac{5}{12}$ at £1 19s. 11¼d.

When there is a fractional part of a penny other than farthings, as

Ex. VII. 4176 at £3 5s. $4\frac{2}{5}$d.

When the quantity, the price of which is required, consists of several denominations, as

Ex. VIII. 9lb. 3oz. 14dwt. at £10 15s. 6d. per lb.

129. Mention was made in (64) of certain fractional parts of £1, 1s., &c., which were termed aliquot parts of £1, 1s., &c. It is advisable to have such parts of the denominations most in use familiarly in the mind; but a pupil will find in PRACTICE, that he has to take aliquot parts of many other quantities which are intermediate between such standard units as £1, 1cwt., &c.: and nothing will render him expert in taking such aliquot parts as he will require, but a readiness in the treatment of fractions.

The following are the most useful aliquot parts of £1 and 1s., and should therefore be remembered.

10s. 0d.	= $\frac{1}{2}$	£
6s. 8d.	= $\frac{1}{3}$	„
5s. 0d.	= $\frac{1}{4}$	„
4s. 0d.	= $\frac{1}{5}$	„
3s. 4d.	= $\frac{1}{6}$	„
2s. 6d.	= $\frac{1}{8}$	„
1s. 8d.	= $\frac{1}{12}$	„
1s. 3d.	= $\frac{1}{16}$	„
1s. 0d.	= $\frac{1}{20}$	„

6d.	= $\frac{1}{2}$	shilling.
4d.	= $\frac{1}{3}$	„
3d.	= $\frac{1}{4}$	„
2d.	= $\frac{1}{6}$	„
1½d.	= $\frac{1}{8}$	„
1d.	= $\frac{1}{12}$	„

In all cases we shall find, that where the price given is not an aliquot part of the unit of the next higher denomination, it is necessary to split the price into two or more portions, of which the largest must be an aliquot part of this said unit, and the remaining portions are aliquot

parts either of this same unit, or of some one of the portions already used in the Ex.

Ex. I. 4108 at 7¾d.

In this Ex. since the price 7¾d. is not an aliquot part of 1s., I therefore break up this price into three parts, 6d., 1½d., and ¼d., of which the largest, 6d. = ½ of 1s. the next higher denomination; the next, 1½d. is an aliquot part of 6d., viz. ¼th; and the last, ¼d. is ⅛th of 1½d. Now, I know that 4108 articles at 1s. each would cost 4108s.; therefore, 4108 at 6d., *i. e.* at ½sh., cost 4108 times ½sh., or ½ of 4108s.: I therefore find the value of this quantity and write it down, viz. 2054s. So also, 4108 at 1½d. = 4108 at ¼ of 6d., and therefore = ¼th of the value of 4108 at 6d., which was found just before. Similarly, since ¼d. = ⅛th of 1½d., therefore 4108 at ¼d. = ⅛th of the previously found value of 4108 at 1½d. Hence the sum of my three amounts at 6d., 1½d., and ¼d., will be the total value, at 7¾d. The whole of the operations which have just been described are to be written out in the accompanying shape.

Ex. I. 4108 at 7¾d.

			4108		
A price 6d.	= ½sh.	gives	2054		
1½d.	= ¼ of 6d.	„	513	6	
¼d.	= ⅛ of 1½d.	„	85	7	
7¾d.		2,0	265,3	1	
			£132	13	1

(See App. Art. Composite Divisor.)

In the second division, 4 being the divisor, I had to find the value of $\frac{2054\text{s.}}{4} = 513\frac{1}{2}\text{s.} = 513\text{s. } 6\text{d.}$; and generally, when as in (64) a number of shillings, or pounds is divided by a divisor, and a rem^{r} is left, the fractional quotient can be converted into positive terms *at once*, and more readily than by the usual method, in which the remainder is reduced to lower den^{n}, and the division again performed. But in the next line, where the divisor is 6, we have to take a sixth part of the 6d. as well as of the 513s., hence I have $\frac{513}{6}\text{ s.} = 85\frac{3}{6}\text{s.} = 85\text{s. } 6\text{d.}$; and this, with the sixth part of 6d., viz. 1d., becomes 85s. 7d. Though it takes a long time to explain these processes,

yet a pupil who is quick in working fractions will obtain the above results more readily than I can describe them, and much time will be saved by their use: but those who prefer the usual method of reducing the remainders, as in Compound Division, can of course adhere to it. The 2653 in the result evidently consists of shillings; and from this Ex. we see that the highest denomination in the sum of all the separate amounts is the same as that of which we took the first aliquot parts.

Ex. II. 4103 at 7s. 5¼d.

			4103		
A price 5s. 0d.	= ¼£	gives	1025	15	
2s. 0d.	= $\frac{1}{10}$£	„	410	6	
4d.	= $\frac{1}{60}$ of 2s.	„	68	7	8
1d.	= ¼ of 4d.	„	17	1	11
¼d.	= ¼ of 1d.	„	4	5	5¾
7s. 5¼d.			£1525	16	0¾

In breaking the price 7s. 5¼d. into parts, we find that 2s. is $\frac{1}{10}$th of £1, and not of the previous aliquot part 5s.; hence, in dividing by 10, we must take as dividend, not the line corresponding to 5s. but the top line, 4103, which is the value of 4103 at £1 each: and in taking aliquot parts, we must always be careful to take as dividend, that line which expresses the value given by that coin or denomination of which we are taking an aliquot part.

Obs. In this and the preceding Ex. I have written the denomination of which the aliquot part has been taken in every line: but for the future I shall generally omit the denomination when I am taking an aliquot part of the line preceding, but insert it when I am taking a part of some earlier dividend.

Ex. III. 6009 at 19s. 5¼d.

			6009		
A price 10s. 0d.	= ½£	gives	3004	10	
5s. 0d.	= ¼	„	1502	5	
4s. 0d.	= $\frac{1}{5}$£	„	1201	16	
4d.	= $\frac{1}{60}$	„	100	3	
1d.	= ¼	„	25	0	9
¼d.	= ¼	„	6	5	2¼
19s. 5¼d.			£5839	19	11¼

This Ex. differs from the preceding, only in having the price above 10s. In such a case we always take 10s. as the first aliquot part; and the remaining shillings and pence as aliquot parts either of 10s. or of any amount which has been used in the course of the example. Sometimes it may happen that a

second aliquot part of £1 is taken, as in 16s. 8d., where I should take 10s. = ½£, and 6s. 8d. = ⅓£.

Ex. IV. 7111 at £1 17s. 4¼d.

Price £1	0s. 0d.	gives	7111	
	10s. 0d. = ½£	„	3555	10
	5s. 0d. = ¼	„	1777	15
	2s. 0d. = $\frac{1}{10}$£	„	711	2
	4d. = ⅓	„	118	10 4
	¼d. = ¼	„	14	16 3¼
£1	17s. 4¼d.		£13288	13 7¼

Here £7111 is the value of 7111 things at £1 each; therefore, if I find the value of 7111 at 17s. 4¼d., as in the last Ex., and add in the top line as £7111, I shall obtain a correct result.

Ex. V. 4013 at £12 7s. 0½d.

			4013	
			12	
A price £12	0s. 0d.	gives	48156	
	5s. 0d. = ¼ £		1003	5
	2s. 0d. = $\frac{1}{10}$£		401	6
	½d. = $\frac{1}{24}$ of 2s.		8	7 2½
£12	7 0½		£49568	18 2½

Here, since £4013 repeated twelve times gives the value of 4013 at £12; therefore I must multiply the 4013 by 12, and add the product as pounds to the other amounts obtained by proceeding with the 7s. 0½d., as in Exs. III. and IV. The last division, by 48, cannot of course be performed mentally: a pupil may obtain the result by Long Division, and merely write down the amount.

Ex. VI. 6583$\frac{8}{15}$ at 1*l.* 19s. 11¾d.

A price £1	0 0	gives	6583*l.*	
	10 0 = ½£		3291	10
	5 0 = ¼		1645	15
	4 0 = ⅕£		1316	12
	10 = ½ of 5s.		274	5 10
	1 = $\frac{1}{10}$		27	8 7
	½ = ¼		13	14 3½
	¼ = ½		6	17 1¾
$\frac{8}{15}$ of 1	19 11¾ = ...		1	1 3$\frac{13}{15}$
			£13160	4 2$\frac{7}{60}$

This Ex. differs from Exs. IV. and V. only in the presence of the fraction $\frac{8}{15}$. I therefore, after having proceeded with the sum, as though the $\frac{8}{15}$ were not there, find the value of $\frac{8}{15}$ at 1*l.* 19s. 11¾d. *i.e.* of $\frac{8}{15}$ of (1*l.* 19s. 11¾d.) which, according to the method of Ex. IV. in (84) = £1 1s. 3$\frac{13}{15}$d.

The value of the $\frac{8}{15}$ might also have been thus obtained:

$$\frac{8}{15} = \frac{5}{15} + \frac{3}{15} = \frac{1}{3} + \frac{1}{5};$$

I might therefore have taken one-third and one-fifth of 1*l.* 19s. 11¾d. and their sum would have amounted to 1*l.* 1s. 3$\frac{13}{15}$d., as before.

Ex. VII. 4176 at £3 5s. $4\frac{8}{10}$d.

				4176		
				3		
A price £3	0	0	gives	12528		
	5	0	= ½£ „	1044		
	4		= $\frac{1}{15}$ „	69	12	
		½	= $\frac{1}{8}$ „	8	14	
		½	= $\frac{1}{80}$ of 4d.	3	9	$7\frac{1}{2}$
		¼	= $\frac{1}{160}$ of 4d.	3	9	$7\frac{1}{2}$
£3	5	$4\frac{8}{10}$		£13657	5	$2\frac{1}{2}$

The presence of the fraction $\frac{8}{10}$d. is the only point in which this Ex. differs from III. and IV.; and just as we break up $\frac{3}{4}$d. into $\frac{1}{2}$d. and $\frac{1}{4}$d. so this $\frac{8}{10}$d. must be broken up into such portions as will be aliquot parts of 1d. viz. $(\frac{4}{10} + \frac{2}{10} + \frac{1}{10})$d., or $\frac{1}{2}$d. + $\frac{1}{4}$d. + $\frac{1}{8}$d.

If in this Ex. there had been a fraction at the end of the 4176, as in Ex. VI., we should have proceeded with it just as with the $\frac{8}{10}$ in that Ex.

Ex. VIII. 9lb. 3oz. 14dwt. at £10 15s. 6d. per lb.

Hitherto we have had the quantities whose value was required expressed all in one den^{n}; and we could therefore repeat the highest den^{n} of the price, as for instance, £1, as many times as there were units in the given quantity, and then take parts of this highest den^{n} for the remainder of the price. But in Ex. VIII. we cannot place 9lb. 3oz. 14dwt. in the top line, and multiply it by £10, because the result would not be £10 repeated an exact number of times: I therefore place the £10 15s. 6d. in the top line, and multiplying it by 9, I obtain the value of 9lb. at £10 15s. 6d. per lb.; and the value of the 3oz. 14dwt. will be found, by taking the same parts of £10 15s. 6d. that 3oz. 14dwt. are of 1lb. Thus working,

				£10	15	6
						9
Value of 9lb.	0oz.	0dwt.	is	96	19	6
	3	0	= ¼ of 1lb. „	2	13	$10\frac{1}{2}$
		10	= $\frac{1}{6}$ of 3oz. „		8	$11\frac{3}{4}$
		2	= $\frac{1}{5}$ of 10dwt.		1	$9\frac{11}{20}$
		2	= $\frac{1}{5}$ of 10dwt.		1	$9\frac{11}{20}$
9lb.	3oz.	14dwt.	=	£100	5	$11\frac{7}{10}$

$$\left(\frac{11}{20} + \frac{11}{20} + \frac{3}{4} + \frac{1}{2}\right)\text{d.} = \frac{11+11+15+10}{20}\text{ d.}$$

$$= \frac{47}{20}\text{d.} = 2\tfrac{7}{10}\text{d.}$$

But since 3oz. 14dwt. $= 3\frac{14}{20}$oz. $= 3\frac{7}{10}$oz. $= \frac{3\frac{7}{10}}{12}$ lb. $= \frac{37}{120}$ lb., therefore the Ex. may be written $9\frac{87}{120}$ lbs. at £10 15s. 6d. per lb.; and it is then similar to Ex. VI.

130. In many Exs. similar in principle to those given above, a knowledge of fractions will enable a pupil to employ very brief methods of working; as, for instance, if I had 317 at 16s. 8d., I should say

317 at 16s. 8d. $= 317$ at $\frac{5}{6}$£ $= \frac{1585}{6}$£ $= 264\frac{1}{6}$£ $=$ £264 3s. 4d.

Again, to find the value of 754 at 7s. 7d.

754 at 7s. 6d. $= 754$ at $\frac{3}{8}$£ $= \frac{2262}{8}$ £ $= 282\frac{3}{4}$£;

$=$ £282 15s.

and 754 at 1d. $= 754$d. $= 62$s. 10d.

therefore 754 at 7s. 7d. $=$ £282 15s. + £3 2s. 10d.

$=$ £285 17s. 10d.

This method is especially worth notice in short Exs.

Thus, 97 at $7\frac{1}{2}$d. $= 97 \times \frac{5}{8}$sh. $= \frac{485}{8}$ sh. $= 60\frac{5}{8}$sh. $=$ £3 0s. $7\frac{1}{2}$d.

We may also observe, that since the cost of 12 things at 1d. each $=$ 1s., therefore that of 12 things at $3\frac{1}{2}$d. $= 3\frac{1}{2}$sh. $=$ 3s. 6d., and of 12 things at $7\frac{3}{4}$d. $= 7\frac{3}{4}$sh. $=$ 7s. 9d.; *i. e.* if I have the price of one article in pence and a fractional part of a penny, the price of 12 articles will be expressed by the same figures as shillings and parts of a shilling.

Hence also the value of any multiple of 12 things may be readily expressed as above, if the price of one be given in terms of pence.

Ex. Find the value of 961bs. at $10\frac{1}{4}$d.

Cost of 121bs. at $10\frac{1}{4}$d. $= 10\frac{1}{4}$s. $=$ 10s. 3d.

therefore cost of 8×121bs. at $10\frac{1}{4}$d. $= 8 \times$ (10s. 3d.) $=$ 82s.

$=$ £4 2s.

With a little practice, such an Ex. as this might be worked mentally, more quickly than it could be written.

APPLICATIONS OF PROPORTION.

RULE OF THREE.

The Articles marked thus (*) may be omitted by those who have not read Fractions.

131.* In articles (65) to (77) the subject of Proportion has been fully discussed: and our business now is, to show how to work Exs., which are in reality only different forms of the question asked in (73), viz. If three numbers be given, what fourth number must be chosen, such that the four, when taken in order, shall be proportionals?

132. The Rule of Three is so called because in the questions given under this head there are three quantities proposed. These three numbers, when placed in order for working an Ex., are called *terms*. In using the expression first, second, and third *terms*, we intend to indicate the manner in which these three given quantities are arranged. Thus, if I were to write 9, 8, 24, as first, second, and third *terms* respectively in the sense here intended, I should have

1st		2nd		3rd
9	:	8	::	24

where the dots placed between the terms are signs which express the relation existing among these terms, so that with them and the fourth term a Proportion is formed. When any three terms are properly arranged at the commencement of an Ex. in Rule of Three they are said to

form a *Statement:* and the question is said to be stated. So long as only three terms are contained in a statement, the question is said to be one of *Simple Proportion:* but sometimes more than three quantities require to be so arranged: the question is then said to come under the head of *Compound Proportion*, or, as it is sometimes called, Double Rule of Three. We shall not, however, enter upon this subject, till we have gone through all the applications of Simple Proportion.

133.* Observing (73) and (74), we learn the following relations between the four terms forming any Proportion.

1st. That the third and fourth terms are of the same kind, *i. e.* that the third term must always be of the same nature as the one required.

2ndly. That the first and second must always be of the same kind; and that they must be reduced to the same denomination, if they be not already so.

3rdly. That the fourth term is obtained by multiplying the third term by the fraction $\frac{2\text{nd}}{1\text{st}}$; or, if this operation be performed at two steps, we multiply by the second term, and divide by the first.

4thly. That if the second term be greater than the first, the fourth term will be greater than the third; but if the second term be less than the first, then the fourth term will be less than the third.

5thly. That since the fraction $\frac{2\text{nd}}{1\text{st}}$ is an abstract number, therefore the fourth term, which $= \frac{2\text{nd} \times 3\text{rd}}{1\text{st}}$, is of the same denomination as that in which the third was expressed.

Collecting together these facts, we deduce the following Rule.

134. RULE. Find out the nature of the quantity sought by the question; *i. e.* of the required fourth term.

Of the three quantities given in the question, find that which is of the same nature as the fourth term, and take that quantity as the third term.

In order to place the other two quantities in their proper situations, inquire whether, from the nature of the question, the fourth term will be more or less than this third term: if *more*, make the *larger* of the two remaining quantities the middle or second term; but if *less*, make the *smaller* the middle term: the only remaining quantity must of course fill the first place.

If the first and second terms be not expressed in the same denomination, reduce them till they become so: and if the third term consist of several denominations, reduce it to the lowest name mentioned.

Then multiply the second and third terms together, and divide the product by the first: the quotient will be the answer or fourth term, expressed in the same denomination as that in which the third term was left. If this quotient be expressed in too low a denomination, [as, for example, 1257 farthings, or 1836 dwts.,] let it be reduced to a higher denomination: [as £1 6s. 2¼d. and 7lb. 7oz. 16dwts.]

135. I will now proceed to work some Exs. which will illustrate the different varieties that may be expected under the head of Simple Proportion: and it will be found that the principal difficulty consists in arranging the three terms according to the directions prescribed by the Rule. This is especially the case, when the question is given in such a shape, that the three terms cannot be *immediately* obtained from the question as it stands. I will explain this more *fully as I go* through the various kinds of Exs. When the

Statement is once obtained, the remainder of the work consists merely of Multiplication, Division, and Reduction.

I recommend a pupil to work every question that I have worked, so that he may the better see the correctness and ascertain the object of the successive operations in any Example.

Ex. I. If 12 yards of cloth cost £19, what will 8 yards cost?

Here I see that the required fourth term will be money; I therefore place the £19 in the third term. Also, the fourth term, which is to be the price of 8 yards, will be *less* than the third term, which is the price of 12 yards; therefore, according to the Rule, I place the *smaller* of the two remaining terms, *i. e.* the 8 yards, in the middle, and the 12 yards in the first place.

yds.	yds.	£
12 :	8 ::	19
		8
	12)	152
		£12 13s. 4d.

The first and second terms are already in the same name, and the third term contains but one denomination, therefore no reduction is required. I now multiply the second and third terms together, and divide by the first: the answer is £12 13s. 4d. And this fourth term and the other three terms form the following proportion:—

12 yds. : 8 yds. :: £19 : £12 13s. 4d.

or in words, 12 yards are to 8 yards, as £19 are to £12 13s. 4d.

Ex. II. If 17cwt. 3qrs. 14lbs. cost £8 18s. 9d. how much may be bought for £5 12s. 6d. at the same rate?

The fourth term will evidently be expressed in weight; therefore I put 17cwt. 3qrs. 14lbs. in the third term. I now ask this question: if the quantity in the third term can be obtained for £8 18s. 9d., will more or less be bought for the £5 12s. 6d.? evidently *less*; therefore I place the *less* of the two prices in the middle, and the remaining one first. The first and second terms are not expressed in any single denomination; I therefore reduce them to threepences, which is the highest denomination to which they can both be reduced. Also, since the third term consists of more denominations than one, I reduce it to the lowest denomination mentioned, viz. lbs. After I have multiplied the second and third terms together, and divided by the first, the quotient is 1260, which consists of lbs., because the third term was expressed in lbs. This quotient, when reduced to higher denominations, becomes 11cwt. 1qr.

£ s. d.		£ s. d.		cwt. qrs. lbs.
8 18 9	:	5 12 6	::	17 3 14
20		20		4
178		112		71
4		4		28
715		450		582
				142
				2002
				450
				100100
				8008

715) 900900 (1260lbs.
715
1859
1430
4290
4290
0

28 { 4)1260
7) 315
4|45

11cwt. 1qr.

I will now give a few Exs. in which the difficulty consists in preparing the question for being stated; but I shall merely show how to overcome the difficulty, and leave the question to be worked out as in the former Exs.

Ex. III. A bankrupt's effects amounted to £980 10s., and he paid his creditors 13s. 4d. in the pound; what was the amount of his debts?

At first sight there appear to be only two terms in this question, but the £1 furnishes another term. Now all the three quantities are money; but by reading the question thus: "If 13s. 4d. be paid for a debt of £1, what debt will be paid by £980 10s.?" I learn that 13s. 4d. and £980 10s. are money *paid*, and the £1 is money *owed, or debt;* and since the fourth term is *debt*, I place the £1 in the third term. Also, the debts which are paid by £980 10s. are of course more than this third term; therefore I place the larger term, £980 10s., in the middle, and 13s. 4d. in the first place. The statement will then be 13s. 4d. : £980 10s. :: £1. The fourth term will be found to be £1470 15s.

136.* The four terms, arranged as a proportion, will be

13s. 4d. : £980 10s. :: £1 : £1470 15s.

or, as two equal ratios—

$$\frac{\text{13s. 4d.}}{\text{£980 10s.}} = \frac{\text{£1}}{\text{£1470 15s.}}$$

which, expressed in words, indicates that the payment, 13s. 4d., is the same portion of the whole payment £980 10s., that the debt of £1 is of the whole debt £1470 15s.

137. Ex. IV. How much may a person spend in 73 days, if he wishes to lay by every year 50 guineas out of an income of £450?

Since I wish to know how much may be spent in 73 days, I must ascertain how much is spent in one year: I therefore subtract the 50 guineas, which are saved, from the whole income of £450: the remainder is £397 10s., and the question now becomes—"If in 365 days I spend £397 10s., how much may I spend in 73 days?" The statement will be

			£	s.
			450	0
days.	days.		52	10
365	: 73	::	397	10

and the required sum will be found to be £79 10s.

Ex. V. If the sixpenny loaf weigh 3lbs. when wheat is 6s. per bushel, what should it weigh when wheat is 6s. 9d. per bushel?

This seems a very simple question, but most pupils make an error in the statement; for, after correctly placing the 3lbs. in the third term,—when they ask the question, as to whether the answer will be more or less than this term, they reason that if the loaf weigh 3lbs. when wheat is at 6s., it will weigh more when wheat is at 6s. 9d.; whereas, since the wheat is dearer, we ought to have a less weight of bread for the same money: and the statement will be

6s. 9d. : 6s. :: 3lbs.

138. For the method of working such Exs. as J 10, see Appendix, Art. Proportion: and for such as S 4 and T 1, see Art. Duodecimals.

139.* I will now by a few Exs. illustrate the use of Fractions.

Taking the statement already made in Ex. III., I have

$$\text{Fourth term} = \frac{\text{2nd} \times \text{3rd}}{\text{1st}} = \frac{(\text{£980 10s.}) \times \text{£1}}{\text{13s. 4d.}}$$

$$= \left(\begin{array}{l}\text{when second and first are re-}\\ \text{duced to a fractional part}\\ \text{of a £.}\end{array}\right) \quad \frac{980\tfrac{1}{2}£ \times 1£}{\tfrac{1}{2}£}$$

$$= \left(\frac{1961}{2} \times 1 \times \frac{3}{2}\right)£$$

$$= \frac{5883}{4}£ = 1470\tfrac{3}{4}£$$

$$= £1470\ 15s.$$

Again, in Ex. V.

$$\text{Fourth term} = \frac{6s. \times 3lbs.}{6\tfrac{3}{4}s.} = \frac{6 \times 3}{\frac{27}{4}}lbs. = \frac{\overset{2}{\cancel{6}} \times \overset{1}{\cancel{3}} \times 4}{\underset{1}{\cancel{27}}} = \frac{8}{3}lbs.$$

$$= 2\tfrac{2}{3}lbs.$$

Ex. VI. If $4\frac{3}{8}$oz. avoirdupois cost $8\frac{1}{4}\frac{1}{2}$s., what will $8\frac{1}{4}\frac{1}{2}$lbs. cost?

$$4\tfrac{3}{8}oz. : 8\tfrac{1}{4}\tfrac{1}{2}lbs. :: 8\tfrac{1}{4}\tfrac{1}{2}s.$$

$$\text{Fourth term} = \frac{8\tfrac{1}{4}\tfrac{1}{2}lbs. \times 8\tfrac{1}{4}\tfrac{1}{2}s.}{4\tfrac{3}{8}oz.} = \frac{\overset{9}{\cancel{99}} \times 16oz. \times \overset{3}{\cancel{99}}s.}{\underset{8}{\cancel{35}}oz.}$$

$$= \left(\frac{\overset{5}{\cancel{285}}}{\underset{8}{\cancel{35}}} \times \frac{\overset{2}{\cancel{16}}}{1} \times \frac{287}{\underset{2}{\cancel{32}}} \times \frac{\overset{3}{\cancel{9}}}{\underset{1}{\cancel{21}}}\right)sh. = \frac{15 \times 287}{16}s.$$

$$= \frac{4305}{16}s. = 269\tfrac{1}{16}s.$$

$$= £13\ 9s.\ 0\tfrac{3}{4}d.$$

Questions wherein fractional quantities occur in the terms may also be worked by the use of Decimals; but this is not an advantageous method of solving them, especially when some of the decimals are circulating. See Ex. IV. in (118).

INTEREST.

140. INTEREST is the payment made for the use of money for any time, and is generally reckoned at so many pounds a-year for £100 lent; or, as it is commonly called, so many pounds per cent. For instance, if £5 be the interest of £100 lent for a year, we should say that the money is lent at the rate of 5 per cent. per annum.

The sum lent is called the *Principal;* the interest of £100 for one year the *Rate;* and the sum lent, together with its interest, for any length of time, is called the *Amount.*

When interest is paid only upon the sum originally lent, it is called *Simple* Interest; but when at the end of any time agreed upon, as for instance a year, the interest is added to the principal, so that this amount forms the principal for the next year; and a similar addition is made at the end of every such period, then it is termed *Compound* Interest.

141. The questions which occur in this Rule are merely Examples of Proportion.

Ex. I. Find the Simple Interest of £382 10s. for one year at 5 per cent.; in other words—If a principal £100 give £5 interest, what interest will be derived from the principal £382 10s.? the statement will evidently be

£100 : £382 10s. :: £5.

Also, it will be found that every Ex. in Simple Interest will furnish a similar statement, in which we observe that the first term is £100, the second is the principal, and the third is the rate.

We now multiply the second term by the 5, and divide by the 100; *i. e.* we multiply by the rate and divide by 100. And in these Exs. we do not, as usual in the Rule of Three, reduce the first and second terms to the lowest denomination expressed in either of them, but multiply by 5 and divide by 100, as in Compound Multiplication and Division. The sum, when worked in the usual form, stands thus:

£100 : £382 10s. :: £5

5

1,00) 19,12 10

20

2,50

12

6,00 *Answer.* £19 2s. 6d.

Here the division by 100 is performed by cutting off two ciphers at the end of the divisor, and two figures at the end of the dividend: and the remainder after each division is reduced, as in Compound Long Division. I have written the 100 as a divisor, but in practice it is omitted.

142. If the interest for any number of years is required, multiply the interest for one year by the number of years. If for any number of months or weeks, aliquot parts of the interest for one year may be taken, as in Practice: but if for any number of days, it should be found by Proportion.

Ex. II. Find the interest of £175 for three years and 135 days, at five per cent.

The interest of £175 for one year is £8 15s., and for three years is $3 \times (£8\ 15s.) = £26\ 5s.$ Now, to find what is the proportionate amount of interest for 135 days, we have this statement;

days.		days.		£	s.
365	:	135	::	8	15

and the answer is £3 4s. 8$\frac{4}{73}$d.: therefore the whole interest = £26 5s. + £3 4s. 8$\frac{4}{73}$d. = £29 9s. 8$\frac{4}{73}$d.

Under the head of Simple Interest may be included all questions generally classed under the heads of Commission, Brokerage, and Insurance; for all such quantities are calculated at a fixed rate for every £100.

143. As an Ex. in Compound Interest we may take the following question:—

What is the amount of £350 in three years at five per cent. Compound Interest?

Putting down only the results of the three operations, we have

	£	s.	d.
Principal of first year	350	0	0
Interest of first year	17	10	0
Amount of first year, and principal of second year..	367	10	0
Interest of second year	18	7	6
Amount of second year, and principal of third year	385	17	6
Interest of third year	19	5	10$\frac{1}{2}$
Amount at the end of third year	405	3	4$\frac{1}{2}$

144.* Since in (141) we multiplied by 5, and divided by 00, therefore we might at once have multiplied by the action $\frac{5}{100}$, or $\frac{1}{20}$: that is, the operation of finding the terest might have been performed mentally by dividing 20: and this is generally done when the rate is 5 per nt., but not otherwise.

145. Questions may be found in Interest which involve me little difficulty, because there do not appear at once ree terms out of which to form a statement. Take for stance

Ex. III. In what time will £75 12s. 6d. amount to £99 16s. 6d. at ur per cent. per annum?

Here the interest gained is found by subtracting the principal, 75 12s. 6d., from the amount, £99 16s. 6d., and it = £24 4s. Also, the terest of £75 12s. 6d. for one year at 4 per cent. is £3 0s. 6d. Hence have this question:

If £75 12s. 6d. produce £3 0s. 6d. in one year, in what time will it oduce £24 4s.? The statement is

£3 0s. 6d. : £24 4s. :: 1 year,

nd the fourth term will be found to be 8 years.

Ex. IV. What sum will amount to £104 2s. 6d. in four years at $4\frac{1}{2}$ r cent. simple interest?

I must here inquire what £100 would amount to in four years at per cent.; the answer is £118. The question is now, therefore,

If £100 become £118, what sum will in the same time amount to 104 2s. 6d? The statement is

£118 : £104 2s. 6d. :: £100

nd the required fourth term is £88 4s. $9\frac{1}{59}$d.

Ex. V. At what rate per cent. will £152 10s. amount to £191 7s. 9d. six years?

I have in this Ex. to find the interest of £100 for one year. Now the terest gained by £152 10s. in six years is £38 17s. 9d., or £6 9s. $7\frac{1}{4}$d. one year: hence the question now is

If £152 10s. gain £6 9s. 7½d. what will £100 obtain? The statement will be

£152 10s. : £100 :: £6 9s. 7½d.

and the fourth term will be found to be £4 5s., *i. e.* the rate per cent. is 4½.

146.* Questions concerning Annuities, Leases, and Reversions involve applications of Interest; but they generally require for their solution either algebraical expressions or tables derived from these. The following Exs. are worth notice.

Ex. VI. What must I give for a freehold, let for £225 a year, so as to have 4½ per cent. for my money? Or in other words,

If every £100 laid out bring £4½, what sum will produce £225? The statement will be

£4½ : £225 :: £100

and the fourth term $= \frac{225 \times 100}{4\frac{1}{2}}$ £ $= \overset{25}{\cancel{225}} \times 100 \times \frac{2}{\cancel{9}}$ £ = £5000.

147.* Sometimes in speaking of the price of a piece of property, it is said that a certain number of years' purchase is given for it: this is the same as so many years' rental. Thus, if a field, the rent of which is £4, be sold for £100, we say that 25 years' purchase was given for it, because the price is 25 times the rental.

Ex. VII. How many years' purchase should be paid for freehold property to clear 4½ per cent?

I must here see how many times a rent of £4½ must be repeated to produce £100, the price of the land which gives £4½.

This number $= \frac{£100}{£4\frac{1}{2}} = 100 \times \frac{2}{9} = \frac{200}{9} = 22\frac{2}{9}$; the price paid is therefore said to be 22⅔ years' purchase.

DISCOUNT.

The reader will find that he cannot proceed beyond this point, without a knowledge of Fractions.

148. DISCOUNT is an allowance made by a creditor to a debtor who pays a debt before it is due. When this allowance is subtracted from the debt, the remainder, *i. e.* the sum that is paid, is called the present worth.

Discount is calculated at a certain rate per cent., and in common usage is treated just the same as Interest: we shall, however, show that this is not strictly correct, but that the person who pays the money has thereby more than the just allowance made to him.

For instance; if £50 were due to me at the end of one year, but I were willing to allow a discount of 5 per cent. for ready money, then according to the common usage, I should throw off the interest of £50 for one year, viz. £2 10s., and receive only £47 10s. But if I make a creditor an allowance for paying ready money, I do so upon the supposition that I can place out to interest the ready money which I receive, and together with the interest can make up the £50 at the end of the year. Now, if I put out to interest £47 10s. at 5 per cent., I shall obtain as interest £2 7s. 6d., and therefore I shall in all receive £49 17s. 6d.: hence I lose 2s. 6d. by this arrangement. The real question now is—What sum put out to interest for a year at 5 per cent. will amount to £50? Or—If

£100 will amount to £105, what sum will amount to £50? The statement will be

£105 : £50 :: £100 (A)

and the fourth term will be found to be £47 12s. 4¼d.

Also, the interest of £47 12s. 4¼d. for one year is £2 7s. 7¾d.; and this, together with the principal = £50: and therefore I neither gain nor lose.

By observing (A) we notice that the third term, £100, is the present worth of the first term, £105; and the fourth term is the present worth of the second term, £50: and in any question where the present worth of a sum is required, the third term is £100; the first term is the amount of £100 at interest for the given time; and the middle term is the sum due.

149. If the discount, and not the present worth, be required, we must place in the third term the discount of £105, viz. £5. But since the discount in the third term would generally require to be reduced to the lowest denomination expressed, and the work be thereby rendered heavy; it is therefore generally better to find the present worth, and then obtain the discount by subtracting the present worth from the bill due.

150. The most common form in which discount occurs is in the use of what are called *Bills*, which are stamped papers, bearing a written engagement to pay a sum of money at some certain future time. If such a bill be presented to a banker before it is due, *i. e.* before the time fixed for payment, and the persons who are responsible for this bill are considered able to meet it at the proper time, the banker will give ready money for it, retaining, however, the *discount* upon the sum, as his remuneration for the accommodation.

In practice, as was said, it is usual to charge interest, and not discount: therefore the banker gains by the transaction, and the amount of this gain will be found to be the interest upon the true discount. For if we refer to the Ex. in (148) we shall see that in discounting a bill of £50 due in six months, the banker would deduct £2 10s., *i. e.* the interest on £50; whereas he ought to have deducted only £2 7s. 4½d., which is the interest upon £47 12s. 7½d.; therefore the extra sum which he takes is the interest upon £2 7s. 4½d., or the interest upon the true discount.

Though discount is not in practice correctly used, yet a pupil in working Exs. should always employ the true method.

The bills mentioned above are said to be *drawn* upon the person or persons who agree to pay the money, and those who allow any such bill to be drawn on them are said to *accept* it: hence they are called acceptors, and the bill itself is called an acceptance. These acceptances are generally for any number of calendar months: but in this country three days, called *Days of Grace*, are allowed after the bill is nominally due, before it is legally due; so that a bill drawn on March 30th, at three months, would not be legally due till July 3rd.

Ex. II. What does a banker gain by discounting a bill of £403 4s., drawn Oct. 13, at four months, and discounted, Dec. 5, at 4 per cent?

Here the bill is legally due on Feb. 16, and from Dec. 5 to Feb. 16 are 73 days:

	£	s.	d.
The interest for that time.. =	3	4	$6\frac{1}{73}$
And the true discount .. =	3	4	0
Therefore the banker's gain =			$6\frac{1}{73}$d.

PROFIT AND LOSS.

151. PROFIT AND LOSS is another application of Proportion. It is calculated at so much per cent., or at a certain *per centage*, and the general object of all Exs. under this head is, to find—(1) What per centage of profit or loss will result from selling an article at a certain price:—(2) At what price it must be sold, that there may arise a certain per centage of profit and loss; the prime cost of the article being in both cases known.

152. It will not be attempted to exhibit an Ex. of every kind of question that might arise; but a sufficient number will be given to show the principles upon which all the questions depend; and the particular method of applying the principles of Proportion in each case must be left to the judgment of the pupil.

Ex. I. If an article cost £2 7s. 3d., and be sold for £3 3s. 0d., what is the gain per cent.?—or, If £2 7s. 3d. become £3 3s. 0d., what will £100 become? The statement is

£2 7s. 3d. : £100 :: £3 3s.

and the fourth term is £133 6s. 8d. Hence the gain upon £100 is £33 6s. 8d.; or the profit is at the rate of 33⅓ per cent.

Ex. II. If, by selling tea at 6s. 4d. per lb., a grocer lose 6 per cent., what was the prime cost per lb.?

Now, to lose 6 per cent. is to obtain only £94 for every £100 laid out; hence the question is really this—If £100 be laid out, and £94 be received for it, what is laid out when 6s. 4d. is received?

Here, since £100 is prime cost, or buying price, and we want buying price in the fourth term, we have this statement:

£94 : 6s. 4d. :: £100

and the fourth term, or prime cost of 1 lb. is 6s. 8$\frac{1}{47}$d.

III. At what price must I sell a commodity purchased at the rate 5s. per cwt. so as to gain 21 per cent.?

is Ex. it is required to receive £121 for £100 laid out; therefore ; the selling price of that which cost £100; and since the fourth to be the selling price of 1cwt., we have

£100 : £14 5s. :: £121

required price is £17 4s. 10½d.

› following Ex. involves the principles of Exs. II. II.

IV. A person, by disposing of goods for £182, loses at the rate of ent.; what should have been the selling price, so as to make a f 7 per cent.

may work this question by two operations: first, find the prime id then from it find the selling price which would give a profit of ent. Since to sell at 9 per cent. loss is to receive but £91 for that ost £100, we have this statement:

£91 : £182 :: £100

answer is the prime cost £200. Also, to obtain a profit of 7 per to receive £107 for that which cost £100; therefore, to find the price of that which cost £200, we have

£100 : £200 :: £107

› answer is £214, the price at which the goods should be sold to per cent. profit.

der to work the question by one statement, we may put it under m—If goods sold for £182 bring £91 for every £100 laid out, ght they to be sold for, so as to bring £107 for every £100?

we have the £100 laid out the same in both circumstances, and nerefore not affect the question: also, £107 is the price that should ved to make 7 per cent. profit; and our fourth term will be the ich should be received: hence the statement is

£91 : £182 :: £107;

fourth term, or selling price, is £214, the same that was obtained › former statements.

PARTNERSHIP.

153. PARTNERSHIP, or as it is sometimes called, FELLOWSHIP, is the Rule by which we determine how to divide profits, which arise from different sums of money put into a business by two or more persons, either for the same or different periods of time.

Ex. I. Two persons enter into business as partners; one puts in £350, and the other £500; they gain £100. How is the profit to be divided?

Here the profit, £100, is made from the whole capital, £850; and each partner's share of the profit will be in proportion to his share of the capital: therefore the question divides itself into these two parts:—(1) If £850 produce a profit of £100, how much will £350 produce? (2) If £850 produce £100, what will £500 produce?

The statements for these two questions will plainly be

	£		£	::	£
(1)	850	:	350	::	100
(2)	850	:	500	::	100

	£	s.	d.
and the fourth term of (1) =	41	3	$6\frac{6}{17}$
" " " (2) =	58	16	$5\frac{5}{17}$
and these together =	100	0	0

Ex. II. A field of grass is rented by two persons for £27; the one keeps in it 15 oxen for ten days, and the other 21 oxen for seven days: find the rent to be paid by each, supposing the pasturage to remain equally good throughout?

Here it is plain that the keep of 15 oxen for ten days is the same as of ten times 15, or 150 oxen for one day: so also, of 21 oxen for seven days is the same as 7 times 21, or 147, for one day; therefore the question is plainly this: If one man turn into a field 150 oxen, and another 147, for one day, and they together pay £27, how is that payment to be divided?

The whole number turned in would be 297, and the two statements would be similar to those in Ex. I., viz.

297	:	150	::	£27	(1)
297	:	147	::	£27	(2)

the fourth term of (1) $= \frac{150 \times 297}{297} £ = £13\ 12s.\ 8\frac{8}{11}d.$
11

„ „ „ (2) $= \frac{147 \times 297}{297} = £13\ 7s.\ 3\frac{3}{11}d.$
11

and their sum $= £27\ 0\ 0$

154. I will give one more Ex. which is the same in principle as Exs. I. and II., but is more complicated in its operations.

Ex. III. On the 1st of January A brought into a business £350, and on the 1st of April £500 more: on the 1st of June he takes out £400; three months after this he brought in £600. B brought into the business £500: four months after this he takes out £150; and on the 1st of November he brought in £650. At the end of the year their clear gain is £1008. How much ought each to receive?

Here A put in £350 from January 1st to June 1st, or five months; also, £500 from April 1st to June 1st, or two months: he has now in the business £850, but he takes out £400, leaving £450. This £450 is in from June 1st to December 31st, or seven months. Also, he has £600 in from September first to December 31st, or four months. Hence he has in all

£			£		
350	for 5	months, or	1750	for 1	month
500	„ 2	„ or	1000	„ 1	„
450	„ 7	„ or	3150	„ 1	„
600	„ 4	„ or	2400	„ 1	„
Therefore he has in all			8300	„ 1	„

Again, B brought in £500 from January 1st to May 1st, or four months: he now takes away £150, and has in £350 from May 1st to December 31st, or eight months. Also, he brings in £650 from November 1st to December 31st, or two months. Hence he has

£			£		
500	for 4	months, or	2000	for 1	month
350	„ 8	„ or	2800	„ 1	„
650	„ 2	„ or	1300	„ 1	„
Therefore B's capital is			6100	„ 1	„
and A's capital was			8300	„ 1	„
therefore the joint capital =			14400	„ 1	„

Hence the two statements will be

	£		£		£
For A's share	14400	:	8300	::	1008
For B's „	14400	:	6100	::	1008

and the fourth terms are £581 for A, and £427 for B.

155. In this place we may introduce an Ex. of the following kind.

Ex. IV. A wine merchant mixes together 20 gallons of wine at 12s. a gallon, 25 gallons at 14s., and 36 gallons at 16s.: what should be the price of a gallon of the mixture?

Here it is plain that	20	gallons at	12s.	are worth	240s.
also „	25	„	14s.	„	350s.
and „	36	„	16s.	„	576s.
and therefore that	81	of the mixture		„	1166s.

hence the value of one gallon is plainly $\frac{1}{81}$th of the value of the whole;

or, price per gallon $= \frac{1166}{81}$ s. $=$ 14s. $4\frac{52}{81}$d.

156. The following Ex. shows how to divide a given quantity into parts which shall have to each other given ratios. It is upon the same principle as the previous Exs., though not commonly recognized as such.

Ex. V. Divide 1065 into parts which shall be to each other in the ratio of 3, 5, 7; and also into parts which shall be in the ratio of $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$.

Taking the former part of the Ex., we observe that $3 + 5 + 7 = 15$ is the smallest integer which can be divided in the ratio of 3, 5, 7; hence the required portions of 1065 will bear the same ratio to 1065 that 3, 5, 7 do respectively to 15. The first portion is obtained by the statement

$$15 : 3 :: 1065;$$

and the fourth term $= 213$. Also, if 5 and 7 be successively placed in the second term of the above statement, we shall find the remaining portions to be 353, and 497.

Similarly, in the latter part of the Ex. we see that $\frac{1}{3} + \frac{1}{5} + \frac{1}{7}$, or $\frac{35 + 21 + 15}{105} = \frac{71}{105}$ is a fraction which can be readily broken up into fractions which shall be in the ratio of $\frac{1}{3}$, $\frac{1}{5}$, and $\frac{1}{7}$; hence the required portions of 1065 will bear the same ratio to the entire number 1065 that

$\frac{1}{3}$, $\frac{1}{5}$, and $\frac{1}{7}$, respectively bear to their sum $\frac{71}{105}$; we have, therefore, as the statement for obtaining the first portion

$$\frac{71}{105} : \frac{1}{3} :: 1065;$$

and the fourth term $= \frac{\overset{5}{\cancel{105}}}{1} \times \frac{1}{\cancel{3}} \times \frac{105}{\cancel{71}} = 525$. So also the second and third statements will be

$$\frac{71}{105} : \frac{1}{5} :: 1065;$$

$$\text{and } \frac{71}{105} : \frac{1}{7} :: 1065;$$

and the corresponding fourth terms will be found to be 315 and 225.

STOCKS.

157. The government of a country sometimes finds it necessary to borrow money; and it gives to the lender a bond acknowledging the debt, and agreeing to pay a certain rate of interest for the money. The amount owing to those who hold these bonds is called the National Debt, or *the Funds;* and the interest paid is derived from the income of the country, arising principally from taxation.

The above-mentioned bonds are saleable, and are called *Stock;* and they of course vary in value, principally according to the plentifulness or scarcity of money.

Thus, suppose a person lend £100 to government, and receive an acknowledgment for it, with an agreement to pay £3 a year interest for the loan; then, if at any time he wishes to sell the bond, and money is scarcer than when he lent the £100, he will get less for it than he gave, perhaps £95; and if money be more plentiful, he will get more for it, perhaps £105. But still the bond represents an acknowledgment for £100, and £3 interest is paid to the

holder of it, whatever may be the sum which he has paid for it. When, therefore, we say that 3 per cent. stock is selling at 80, we mean that the buyer of £100 bond, or as it is called, £100 stock, has to give only £80 sterling for it; hence, as he gets £3 interest for it, it is not £3 *per cent.* to him, but £3 for £80. From this it is plain that the lower the stock is in price, the better interest the buyer obtains: and the higher the stock is, the less interest he obtains for his money.

When the market price of £100 stock is exactly £100, the stock is said to be at *par;* if the price be more than £100, it is said to be at a *premium;* if below £100, at a *discount.* The smallest variation in the price of stock is one-eighth of £1, or 2s. 6d., for every £100 stock.

158. Since persons who wish to sell stock may not know any who wish to buy, therefore all sales and purchases are transacted through agents, who are called Stock-brokers. The broker of the buyer deals with the broker of the seller, and each charges his employer, or principal, as he is called, a commission of $\frac{1}{8}$th of £1 for the transfer of every £100 stock; so that a buyer must always consider that he pays $\frac{1}{8}$th per cent. more, and the seller that he receives $\frac{1}{8}$th per cent. less than the selling price: but if in any Ex. the commission be not mentioned, no notice need be taken of it.

159. In working the various questions that occur in Stocks, a pupil must be careful not to confound stock and actual money. Also, in buying or selling stock, it is quite immaterial whether it be 3 per cent., 4 per cent., or any other kind of stock, unless we wish to know the *income* to be derived: for instance, if I have to sell out £100 stock, when the price is 95, it matters not whether it be in the *3, 4, or 5* per cents.: I have to receive £95 for every £100.

Ex. I. What must be given for £5050 stock in the Three per Cents., at 85⅝ per cent?

Here the price of £100 stock is £85⅝; and I have to find the price of £5050 stock: hence the statement will be

£100 : £5050 :: £85⅝

Multiplying by 85⅝, after the method of (84), and then dividing by 100, we have the annexed operation, in which we find the cost of £5050 stock at 85⅝ = £4311 8s. 9d.

		£
		5050
		85⅝
		25250
		40400
⅝, or ½ of £5050	=	1262 10
¼ of £5050	=	631 5
1,00)		4311,43 15
		20
		8,75
		12
		9,00

Next, let us find what amount of stock any given sum will buy, when the price of £100 stock is known.

Ex. II. How much stock can be bought for £1490, the price being 88⅛, and commission ¼ per cent?

Adding the commission to the market price, we have the cost to the buyer 88⅜, or 88½. The term sought is amount of stock; and in the proposed question £100 is the stock to be bought by £88½: we have therefore

£88½ : £1490 :: £100

and the fourth term, or amount of stock bought by £1490 is £1683 12s. 3⅓d.

Ex. III. If a person invest £2000 in the Three per Cents. when they are at 95¼, what is his annual income therefrom?

In this case the buyer gives £95¼ for £100 stock, *i. e.* for the privilege of receiving £3 interest: hence the question is,—If £95¼ produce £3 interest, what will £2000 produce? Our statement is

£95¼ : £2000 :: £3

and the fourth term, or interest of £2000, will be found to be £62 16s. 6⅓⅓d.

Ex. IV. Find what per centage will be obtained by investing in the Three and a-half per Cents. at 91: or, in other words,—If £91 give £3½, what will £100 produce? The statement is

$$£91 \;:\; £100 \;::\; £3\tfrac{1}{2}$$

and the fourth term $= \frac{3\frac{1}{2} \times 100}{91} £ = \frac{7}{2} \times \frac{\overset{50}{100}}{91} £ = 3\frac{11}{13} £$; and transferring the 100 from the left-hand numerator to the right-hand denominator, I have $\frac{3\frac{1}{2}}{91} = \frac{3\frac{11}{13}}{100}$.

Observing this equation, I notice that the right-hand side gives the ratio of the interest of £100 to £100; and the left-hand side gives the ratio of the interest on £100 *stock* to the price of that stock. Now, in finding the per centage which any interest produces, we wish to know what is the ratio of the interest on £100 to £100; and since the right-hand fraction gives this ratio, therefore the former fraction also gives it: hence this fraction gives a standard, by which we can compare the value of the per centages derived from any two investments in different kinds of stock. Thus, if I wish to know whether it will be more advantageous to invest in the Four per Cents. at 95, or in the Three per Cents. at 85, I must compare the fractions $\frac{4}{95}$ and $\frac{3}{85}$.

By (50), the difference is $\frac{4 \times 85 - 3 \times 95}{95 \times 85} = \frac{340 - 285}{95 \times 85}$. Hence, since $340 > 285$, therefore $\frac{4}{95} > \frac{3}{85}$; or an investment in the Four per Cents. at 95 will produce better interest than in the Three per Cents. at 85.

160. We will now give an Ex. combining two or more of the operations exhibited in the previous Exs.

Ex. V. A person transfers £1000 stock from the Four per Cents. at 90 to the Three per Cents. at 72; find how much of the latter stock he will hold, and the alteration made in his annual income.

The first part of the question may be thus expressed: "If a certain sum of money will buy £1000 stock at 90, how much can be bought when the stock is at 72?" The statement will be

$$£72 \;:\; £90 \;::\; £1000$$

and the fourth term $= \frac{\overset{125}{1000} \times \overset{10}{90}}{\underset{8}{72}} £ = 1250£.$

To find the income derived from the £1000 stock and from the £1250 stock, two simple statements might be employed: but where, as in this case, the stock consists of £100 shares, we can work more briefly thus:—£1000 stock = 10 cents.; and since each cent produces £4, the whole 10 produce £40. Also, £1250 stock = $12\frac{1}{2}$ cents.; and since each cent of this stock produces £3, the whole produce $12\frac{1}{2} \times 3£ = \frac{25}{2} \times 3£ = \frac{75}{2}£$ = £37 10s.; hence the difference of the incomes from the two investments = £40 − £37 10s. = £2 10s.

EQUATION OF PAYMENTS.

161. If a person owe another several sums to be paid at different times, and it is required to know at what time it would be just to pay the whole at one payment, this would be a question to be solved by a Rule called Equation of Payments.

To ascertain the method of finding this time of payment, called the equated time, let us take a simple Example.

Ex. I. If £100 be due at the end of six months, and £200 at the end of twelve months; find when it is just to pay the whole in one sum.

It is quite clear that the time of payment will be at some period between the two fixed times, six months and twelve months; hence the former sum, £100, will be paid after it is due, and the latter sum, £200, before it is due.

Now, a person keeping the £100 beyond the appointed time ought, of course, if that were the only money to be paid, to pay interest for it; but, instead of paying interest, he is to make up for the privilege of keeping the £100 by paying the £200 before it is due: it is hence quite clear that he must pay this £200 such a time before it is due, that the interest of the £200 for that time shall just balance the interest he might obtain by keeping the £100 after it was due. The question then really is—How soon will the interest upon £200 produce the same as the interest upon £100? The answer evidently is, in half the time: *i.e.* the time of paying the £200 must be *earlier* than its original time of twelve months, by half as much as the time of paying the £100 is *later* than its original time of six months; therefore, if the payer keep the £100 *four* months beyond the six months, and pay the £200 *two* months sooner than the twelve months, the interest gained in the one case and lost in the other will be just balanced; and the whole sum will have to be paid in ten months.

162. Putting the question in another form, we may consider that when the sums were to be paid at different times, the payer had the £100 in his hands six months, *i. e.* he had the interest of £100 for six months, or of £600 for one month: also, he had the interest of £200 for twelve months, or of £2400 for one month; therefore he had in all the interest of £2400 + £600, or of £3000, for one month. If, then, the debtor has to pay the £300 in one sum, how long ought he to keep it, so that its interest shall equal the interest of £3000 for one month? The answer to this question will be obtained from the following statement

£300 : £3000 :: 1 month;

and the fourth term will be ten months, the same result as before.

163. The truth of the above method has been disputed by arithmeticians upon this ground. Referring to the above Ex., it is said, that since the £100 is paid after it is due, the payer should pay the interest for the time that it is kept back: but since the £200 is paid before it is due, discount only should be allowed thereon: and since the discount is less than the interest, it is said that the payer, by the above method, receives as much more than his due as the interest of £200 exceeds the discount; hence, to make the payment perfectly correct according to this view, we ought to place the payment earlier than ten months. But since we have shown, by the working of the Ex. given above, that the interest gained in one case and lost in the other is equal, we shall, by placing the time for payment of the £300 earlier, rob the payer in two ways—1st, by depriving him of the interest of the £100 during the latter part of the four months that he ought to be allowed

to hold it: and 2ndly, by making him put the £200 into the hands of his creditor for a longer time than two months.

The fallacy of the reasoning which would place discount instead of interest in the question may be shown thus. It is here assumed that discount, not interest, is applicable to all cases where money is to be paid before it is due. Now, this is quite true where only one payment is to be made; for in that case we have laid it down, that the debtor is to pay such a sum as put out to interest shall just amount to the sum due: therefore, in all cases of discount the debtor pays less than he owes. But this is not the case in Equation of Payments: for here the creditor receives the whole of the latter portion of the debt (say £200) *though it be before it is due*, and he can put the *whole* out to interest, which he cannot, in real questions of discount; and the question now is,—not how much must be put out to interest to raise the £200,—but how long must the whole £200 be put out, to raise an amount of interest equal to that which the creditor has lost, by allowing the first payment (say £100), to remain in the debtor's hands beyond its time. The question considered in this view has been satisfactorily answered in the Ex. worked above.

Ex. II: A person owes £800; £200 to be paid in three months, £100 in four months, £300 in five months, and £200 at six months: if the whole were to be paid at once, what would the time of payment be?

Here, by the agreement, the debtor has the interest of

£				£		
200	for 3	months,	or	600	for 1	month
also of 100	„ 4	„	or	400	„ 1	„
also of 300	„ 5	„	or	1500	„ 1	„
lastly, of 200	„ 6	„	or	1200	„ 1	„
800				3700		

therefore he has altogether the interest of £3700 for one month; and the

whole sum to be paid is £800: hence, we have to find in how many months £800 will produce as much interest as £3700 in one month. The result is obtained from the following statement:

£800 : £3700 :: 1 month;

and the fourth term $= \frac{3700}{800} \times 1$ month $= 4\frac{1}{8}$ months.

In the statements which are used in Exs. I. and II. we observe that the third term is one month: hence, since this will in every Ex. be the third term, we shall merely have to divide the second term by the first, in order to obtain the fourth term: also, the first term is the whole amount due; and the second is the amount of all the several sums, when each one is multiplied by its time of payment.

COMPOUND PROPORTION.

164. It was observed in (132) that a question was classed under the head of Compound Proportion, when there were more than three quantities which required to appear in the statement. The following is an Example.

Ex. I. If 12 yards of cloth, 3 quarters wide, cost £19, what will be the cost of 8 yards, 5 quarters wide?

If the width of the two pieces of cloth were the same, we should take no account of this width, whatever it might be: and the question would then become Ex. I. in (135), or one of Simple Proportion. Referring to that Ex. we have the statement

12 yards : 8 yards :: £19 (B)

and the cost of the new piece $= \frac{8 \text{ yds.}}{12 \text{ yds.}} \times £19 = \frac{8 \times 19}{12}$ £.

We will now take into account the two breadths, 3 qrs. and five qrs. and put the following question—"If a piece of cloth cost $\frac{8 \times 19}{12}$ £, when 3 quarters wide, what will it cost when the width is 5 quarters?" The statement would be

3 qrs. : 5 qrs. :: $\frac{8 \times 19}{12}$ £ (C)

and the fourth term $= \frac{8 \times 19}{12}$ £ $\times \frac{5 \text{ qrs.}}{3 \text{ qrs.}} = \frac{8 \times 19 \times 5}{12 \times 3}$£.

Now, if the whole of statement (B), and the first and second terms in (C), be converted into one statement, as follows—

$$\begin{matrix} 12 \text{ yds.} \\ 3 \text{ qrs.} \end{matrix} : \begin{matrix} 8 \text{ yds.} \\ 5 \text{ qrs.} \end{matrix} :: 19£ \qquad (D)$$

and we take the product of the two quantities which stand first, as our first term, and the product of the two in the middle, as our second term, we should have

$$\text{the fourth term} = \frac{8 \text{ yds.} \times 5 \text{ qrs.} \times 19£}{12 \text{ yds.} \times 3 \text{ qrs.}} = \frac{8 \times 5 \times 19}{12 \times 3}£,$$

which is precisely the same as was obtained from the two successive statements: hence such a statement as (D) will produce a correct result. Also, in forming this statement, independently of (B) and (C), I select for the third term that which is similar to the required fourth term, as in Simple Proportion: and in placing the remaining terms, I take each pair separately, and ask the usual question with the third term, as to whether the answer will be more or less than this term; and I arrange this pair precisely as though they were the only two terms which I had to consider. However many pairs of terms occur in the question, they must all be treated in like manner; for the same proof that has shown how to combine the first and second statements, will show how to combine the third with the result of the first pair.

I will work another Ex., and mention the mental operations which must be performed, in order to enable me to place each pair of terms correctly.

Ex. II. A field, 300 yards long and 280 broad, was ploughed by six horses in two days of eight hours each; how many horses will plough a piece of ground 500 yards long, and 315 broad, in three days of ten hours each?

The fourth term will be horses: I therefore place the six horses in the third term. Also, the new field is longer than the one which required six horses; hence, considering the effect of this pair of terms alone, it will require more horses, and I place the larger term, 500, in the middle: so also the second field is broader than the first, and therefore will take more horses, and I place 315 in the middle. Again,

$$\begin{matrix} 300 \text{ yards} \\ 280 \text{ yards} \\ 3 \text{ days} \\ 10 \text{ hours} \end{matrix} : \begin{matrix} 500 \text{ yards} \\ 315 \text{ yards} \\ 2 \text{ days} \\ 8 \text{ hours} \end{matrix} :: 6 \text{ horses}$$

the first field was ploughed in two days, but the new one in three days; hence, since the time is longer, we shall, so far as this pair of terms is concerned, require fewer horses; and the smaller term, two, is to be in the middle. Also, in ploughing the first field, the days were eight hours, but for the second field they are ten hours; hence, with this extra time, fewer horses will be required, and I place the eight in the second place. The whole statement is as annexed; and

$$\text{the fourth term} = \frac{\overset{5}{\cancel{500}} \times \overset{9}{\cancel{3}} \times \overset{2}{\cancel{15}} \times 2 \times \overset{4}{\cancel{8}} \times \overset{2}{\cancel{8}}}{\underset{3}{\cancel{300}} \times \underset{35}{\cancel{280}} \times \underset{2}{\cancel{3}} \times \underset{2}{\cancel{10}}} \text{ horses}$$

$$= 3 \times 2 \text{ horses} = 6 \text{ horses};$$

precisely the same number as before: *i. e.* the increased size of the field, and the increased length of time allowed for the work, are so balanced, that the same number of horses as before is sufficient.

In arranging the several pairs of terms in the statement I seem to be trying at one time to obtain a smaller term, and at another time a larger term than the third; and it is true that some conditions of the question tend to make the fourth term less than this third, and some to make it more: each pair will produce its own effect in increasing or diminishing the required term; and we shall therefore find the result more or less than the third, according as the conditions in the question which would make it more, predominate, or not, over those which would make it less; *i. e.* according as the product of all the terms in the second place is more or less than the product of all in the first place.

It must be carefully observed, that if any pair of corresponding terms be not expressed in the same denomination, they must be so reduced, just as in Simple Proportion, before we commence forming the fraction which will give the fourth term.

EXCHANGE.

165. EXCHANGE, in its simplest meaning, is merely the conversion of any sum of money from the coinage of one country to that of another. To perform this conversion, we must of course know for how much a certain coin of one country can be exchanged in coins of another: for instance, if I wish to exchange a sum of English money for French, I must know how many of the current coins of France, viz. *francs*, will be given me for £1 of English money. This standard of Exchange between two countries is called the *Course of Exchange*, but it is not always the same; and we shall presently show the cause of the variation.

Ex. I. Exchange £850 sterling for francs at 25 francs 15 centimes, or cents; or, in other words,—If £1 sterling can be exchanged for 25 francs 15 cents, what number of francs can be obtained for £850? To obtain this number, I have the following statement:

£1 : £850 :: 25 francs 15 cents.

Also, since 100 cents make one franc, therefore cents may be expressed as decimal parts of a franc; *i. e.* 25 francs 15 cents = 25·15 francs;

$$\text{hence, the fourth term} = \frac{£850 \times 25{\cdot}15}{£1} \text{ francs}$$

$$= (850 \times 25{\cdot}15) \text{ francs}$$

$$= 21377{\cdot}5 \text{ francs}$$

or 21377 francs 50 centimes.

Ex. II. How many pounds sterling can be obtained for 8457 marks 15½ schillings, Hamburg, at the rate of 13 marks 12 schillings for £1 sterling? The statement is

13 mks. 12 sch. : 8457 mks. 15½ sch. :: £1

By Tables of Hamburg Coinage, we find that 16 schillings = 1 mark: hence, reducing the first and second terms of the above statement to half schillings, we obtain by the usual process the fourth term = £615 2s. 6d.

166. The method of these two Exs. will enable us to convert any sums from the currency of one country to that

of another, when the course of exchange between the two countries is known. But we must also be able to perform this conversion between two foreign countries.

Ex. III. Change 1932 florins at Amsterdam for ducats at Naples, the course of Exchange being 80½ florins for 40 ducats. The statement is

80½ florins : 1932 florins :: 40 ducats.

and the fourth term will be found to be 960 ducats.

We now proceed to explain a further and more comprehensive meaning of the term Exchange.

In commercial transactions between different countries it is not usual to pay for goods imported, in coin, or as it is sometimes called, in *specie* or *bullion:* and for two reasons —first, the quantity of foreign goods imported by a country like England is so great, that if paid for in coin, the payment would speedily drain all the coin out of the country, and business could not be carried on. Secondly, there would be the probable loss of the coin by wreck or otherwise in the transmission; besides that there would arise a loss of interest on the money while it was being sent to its destination.

We shall now show how these difficulties may be avoided, when we are dealing with countries which send goods as well as receive them, *i. e.* which export as well as import.

167. The following is a simple Ex. of the manner of conducting these transactions.

Suppose A and B to represent two merchants in America, and C and D two others in England: let C buy of A a thousand pounds worth of goods, and therefore owe him £1000: so, also, let B owe D £1000 by a similar purchase; then, if these sums be paid in coin, £1000 must cross the Atlantic twice.



But, since C has to pay £1000 to A, he would as readily pay it to D in England, if by such payment he could get rid of his liability to A: so also B would pay A, if he could be rid of his debt to D. This simple transaction might, therefore, be completed thus:

Let C send A a bill acknowledging the debt of £1000, and promising to pay £1000 to any one in England who may present the bill to him at the expiration of a certain time: A then sells to B this bill, and, receiving £1000 for it, has no longer any claim upon C. B now sends this bill to D, and D uses it as a bill for £1000, until the expiration of the time named on the bill, when the money is paid by C. Thus these four merchants have been able to have commercial dealings to the amount of £1000 each, without any coin having left either country.

Of course the value of this bill for £1000 depends entirely upon the ability of C to *meet* it, that is, to pay the money at the expiration of the time agreed upon in the bill: and we often find that C, who was considered able to pay at the time he gave the bill, has become a bankrupt before the time of payment; hence the loss falls on D. And this explains the reason why, in a commercial country like England, the failure of one merchant, or firm, causes others to fail: for, in the case above, D may also be liable for bills as well as C; and if not able to obtain the money which he expected from C, may himself become a bankrupt; and so in turn cause other merchants the same loss which he is himself suffering from the failure of C.

Now, there are thousands of merchants in the situation of A and B in America, and similarly of C and D in England. Hence there are, as a general rule, merchants wanting to buy bills, and others wanting to sell them, in

both countries: and what has been said concerning England and America is true with respect to any two countries which export to and import from one another.

These purchases and sales of bills are, for the reason mentioned in (158), conducted through the medium of Bill-brokers.

168. We have just now been supposing that the two countries have imported and exported goods to an equal amount from one another. But suppose that goods had been sent from America to England to a greater amount than from England to America, for instance, to the extent of £1,000,000, then more bills to that amount would go from England to America than from America to England. The merchants in America have in this case plenty of bills, and of course want money for them: but as there are more persons wishing to sell bills than to buy, therefore the bills fetch a lower price: on the contrary, as bills in England are not so plentiful, there are more buyers than sellers, and the bills fetch a higher price than usual. When this increased price exceeds the cost of insurance and loss of interest upon coin sent over to America, the English merchant prefers sending coin instead of paying the increased rate for a bill: and by successive exportations of bullion, the balance of £1,000,000, which was against us, will be paid off: but this increased price of a bill, (which swallows up the profits of business,) or the alternative of paying in coin, causes merchants to be slow in importing until our exports have increased and helped to restore the balance. Here, also, whatever has been said concerning England and America, is of course equally applicable to any two countries which have commercial transactions with each other. In those countries, as South America, where gold and silver

are amongst their principal productions, bullion is as much a regular article of export, as woollen or cotton goods would be from England.

169. If the price of a bill in England, entitling the holder to receive gold in a foreign country, be less than the usual course of exchange, the exchange is said to be in favour of England.

DEF. The state of the exchange between any two countries at any particular period is termed the *Par of Exchange*, or the *Arbitration Price*: also, a *Bill on London*, means a paper entitling the holder to obtain gold in London to the value of the amount mentioned in the bill.

Arbitration is called *Simple*, or *Compound*, according as there are three or more places concerned.

Ex. IV. Bills on Amsterdam, bought in London at 12 florins 14 cents per £ sterling, are sold in Paris at $57\frac{1}{2}$ florins for 120 francs: what is the rate of Exchange between London and Paris?

My object here is to express £1 in terms of francs. Working fractionally, I have

$$£1 = 12 \text{ florins } 15 \text{ cents}$$

$$\text{also, } 57\tfrac{1}{2} \text{ flor.} = 120 \text{ francs}$$

$$\text{therefore 1 flor.} = \frac{120}{57\frac{1}{2}} \text{ francs}$$

and therefore £1 = 12 flor. 15 cents

$$= 12{\cdot}15 \text{ florins} = \frac{12{\cdot}15 \times 120}{57\frac{1}{2}} \text{ francs}$$

The process may also be represented as follows:

$$£1 = 12{\cdot}15 \text{ florins} \qquad \text{(E)}$$

$$\text{and } 57\tfrac{1}{2} \text{ flor.} = 120 \text{ francs} \qquad \text{(F)}$$

therefore, taking the product of the two quantities in the left-hand of (E) and (F), and the corresponding product of the two on the right-hand, we have

$$£1 \times 57\tfrac{1}{2} \text{ flor.} = 12{\cdot}15 \text{ flor.} \times 120 \text{ francs};$$

or, dividing both sides by $57\frac{1}{4}$ flor.

$$£1 = \frac{12{\cdot}15 \text{ flor.} \times 120 \text{ francs}}{57\frac{1}{4} \text{ flor.}}$$

$$= \frac{12{\cdot}15 \times 120}{57\frac{1}{4}} \text{ francs, by (66)}$$

$$= 25 \text{ francs } 35\tfrac{1}{19}\tfrac{3}{9} \text{ cents.}$$

Observing equations (E) and (F) I notice that I commence (E) by placing on the left-hand side the standard coin of one of the two countries, the arbitrated price between which I wish to find: also the left-hand side of (F) is expressed in the same coin as the right-hand of (E); and in like manner we must proceed with any number of equations, till the right-hand side of the last will be expressed in the coin of the second of the two countries which we wish to connect, as in the following question.

Ex. V. A bill upon Hamburgh is bought at 13 marks $10\frac{1}{4}$ schillings per £ sterling, then sold at Amsterdam at $35\frac{3}{4}$ florins per 40 marks: if the proceeds are then remitted to Paris in French bills at $57\frac{1}{4}$ florins per 120 francs, what rate of exchange is there between London and Paris?

$$£1 = 13\text{m. } 10\tfrac{1}{4} \text{ sch.} = 13\frac{10\frac{1}{4}}{16} \text{ marks} = 13\tfrac{1}{4}\tfrac{1}{4} \text{ m.}$$

$$40 \text{ m.} = 35\tfrac{3}{4} \text{ flor.}$$

$$57\tfrac{1}{4} \text{ flor.} = 120 \text{ francs}$$

therefore £1 × 40 marks × $57\frac{1}{4}$ flor. = $13\frac{1}{4}\frac{1}{4}$ m. × $35\frac{3}{4}$ flor. × 120 francs; and dividing both sides of the equation by the product, 40 m. × $57\frac{1}{4}$ flor., we have

$$£1 = \frac{13\frac{1}{4}\frac{1}{4} \text{ m.} \times 35\frac{3}{4} \text{ flor.} \times 120 \text{ francs}}{40 \text{ m.} \times 57\frac{1}{4} \text{ flor.}}$$

$$= \frac{13\frac{1}{4}\frac{1}{4} \times 35\frac{3}{4} \times 120}{40 \times 57\frac{1}{4}} \text{ francs, by (66)}$$

$$= 25 \text{ francs } 55\tfrac{1}{1}\tfrac{9}{6}\tfrac{2}{4} \text{ cents.}$$

We will just give an Ex. of the form in which the state of the exchanges between England and other countries is generally reported in the public prints. The following is from the *Times* of March 22nd, 1848, under the head "Money Market and City Intelligence."

"The last quotation of gold at Paris was about 30 per mille premium, which would give an exchange of 25·91. The last quotation of short bills on London being 26·50, the price of gold would appear to be about $2\frac{1}{4}$ per cent. higher in London than in Paris."

Here a comparison is instituted between the *nominal* rate of exchange of £1 for francs, and the *real* rate at the particular time mentioned. Thus, if 1000 represent the price of gold in Paris when £1 sterling is worth about 25·15 francs, what will be the value of this £1 in francs when 1030 represents the price of gold in Paris? We have, of course, this statement:

$$1000 : 1030 :: 25{\cdot}15 \text{ francs}$$

$$\text{the fourth term} = 25{\cdot}90 \text{ francs}$$

that is, £1 will in Paris produce 25·90 francs. But a person in Paris wishing to buy a bill on London, entitling him to receive £1 in gold, must give 26·50 francs. The difference, ·6 francs, is the amount by which £1 sterling in London is dearer than in Paris.

Also, since 26·50 francs = £1,

$$\text{therefore 1 franc} = \frac{1}{26{\cdot}5}\,£;$$

$$\text{and this extra price of £1} = {\cdot}6 \text{ fr.} = {\cdot}6\,\frac{1}{26{\cdot}5}\,£;$$

$$\text{and therefore of £100 it} = 100 \times {\cdot}6\,\frac{1}{26{\cdot}5}\,£.$$

$$= \frac{60}{26{\cdot}5}\,£ = \frac{12}{5{\cdot}3}\,£ = 2{\cdot}26\ldots.£$$

or $2\frac{1}{4}$ £ nearly.

That is, the price of gold in London is $2\frac{1}{4}$ per cent. greater than in Paris; or a bill which would entitle a person to receive £100 in gold in London would cost £$102\frac{1}{4}$ in Paris.

AREA AND VOLUME.

DUODECIMALS.

170. If any line be taken, as ab, and upon it a square, $abcd$, be described, this figure may be called the square of ab. And if we take ab as an unit of length, viz. 1 inch, 1 foot, &c., then $abcd$ is called 1 square inch, 1 square foot, &c. We have here, therefore, inches, feet, &c. of length, or *linear* inches; and inches, &c. of surface, or *superficial* inches, or square inches.

FIG. 1.

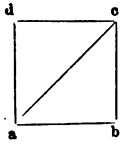
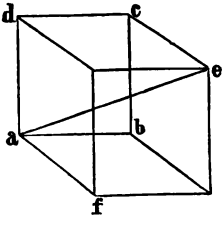


FIG. 2.



171. In like manner, if upon ab, the annexed figure $abcdef$ be described, having six sides, or surfaces, each equal to $abcd$, it is called a *cube*: and, as before, if ab be taken as representing 1 inch, 1 foot, &c., this figure will represent 1 cubic inch, 1 cubic foot, &c.

We have now, therefore, three kinds of units of measurement, viz. linear or common inches; square, or superficial inches; and cubic, or solid inches. Also, these three units are said to contain 1, 2, and 3, dimensions respectively. For example, the floor of a room, having the dimensions length and breadth, is of the same nature as a square, and any such surface is called an *Area*: but a cistern of water, having the three dimensions of length, breadth, and depth, is of the same nature with a cube.

The quantity contained by such a cistern or similar figure is termed its *Volume*, or solid content.

DEF. A surface which is so enclosed by lines that any two which meet in a point are perpendicular to one another, is called *rectangular*.

OBS. Any surface bounded by straight lines may be denoted by two letters placed at its opposite corners; and any solid contained by such surfaces may be so denoted: but the two letters employed should not be joined together by a line. Thus, in Fig. 1, I should say the area bd, not ac; and in Fig. 2, I should say, the volume fc, not ae.

DEF. The lines ac, ae, are called *diagonals*.

172. I have now to explain how to find the area of surfaces and the volume of solids: but as I do not propose to enter upon mensuration generally, I shall merely treat of rectangular surfaces, as squares and oblongs; and of solids, the surfaces of which are also rectangular.

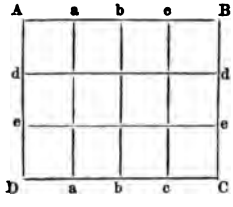
173. Quantities which can be accurately represented by numbers, whether whole or fractional, are called *commensurable*; and those which cannot be so represented, are called *incommensurable*. Thus, it will be shown under the head of "Square Root," that if in Fig. 1 the length of ab be represented by 1, ac cannot be accurately represented by any number whatever, whether whole or mixed. The quantity used for the value of ac is 1·4142.... and this being nearly the true value, is termed its *approximate* value.

174. We may here state as a fact, that the area of any rectangular surface is found by multiplying the numbers which represent its length and breadth; also, that the volume of a solid, bounded by rectangular surfaces, is found by multiplying together the numbers representing the length,

breadth, and depth, or thickness. The above statements are true, whether the lines bounding the area or volume be commensurate or not; but they cannot be proved to be universally true without the aid of geometry. We shall, however, give an Ex. illustrating the correctness in each case, choosing of course only commensurable numbers.

175. Let $ABCD$ be a rectangular figure whose sides, AB and AD, meeting in the point A, and called *adjacent* sides, contain an exact number of units,—viz. $AB = 4$ inches, $AD = 3$ inches: let the opposite sides, AB, CD, be divided into four equal parts in a, b, c, and the sides AD, BC, be divided into three equal parts, in d, e; let the lines aa, bb, cc, dd, ee, be drawn: the figure will be divided into equal squares, each of which has for its side 1 linear inch, as Aa, and is therefore a square inch. Also, counting vertically, there are three rows of squares, because $AD = 3$ inches, and each row contains four squares, because $AB = 4$ inches: and there are in all 12 squares, *i.e.* 3×4 squares: hence we see that the number of square inches in the area $ABCD$ = the product of the number of linear inches in the two adjacent sides, AB, AD.

Fig. 3.



Def. The lines aa, bb, which are drawn so that they are at an equal distance from one another, are called *parallel* lines; so also they are said to be *parallel* to AD or BC.

Hence, if I wish to draw lines, as aa, bb, or cc, in the above figure, I should say—through a, b, c, draw lines parallel to AD or BC. So also, dd, ee, are drawn parallel to AB or CD.

If AD had been $= AB$, then the figure would have been a square, and the number of square inches in it $= 4 \times 4$, or $4^2 = 16$; *i. e.* the number of square inches in any square figure whose side is expressed in linear inches, is found by multiplying the number contained in the side by itself. Hence the second power of a number is called its square, because it represents the area of a square figure, the side of which is the number itself.

We can now show how the numbers mentioned in what is called "Square Measure," are obtained.

For if 12 linear inches = 1 linear foot,
therefore 12×12, or 144 square inches = 1 square foot,

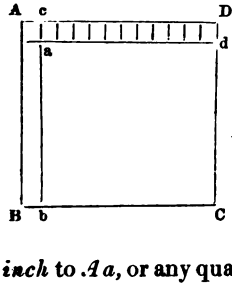
So also, since 3 linear feet = 1 linear yard,
3×3, or 9 square feet = 1 square yard.

Again, $5\frac{1}{2}$ linear yards = 1 linear pole, or perch;
therefore $5\frac{1}{2} \times 5\frac{1}{2}$, or $30\frac{1}{4}$ square yards = 1 sq. perch.

and these square perches, yards, &c., are the quantities always made use of in measuring land; for no amount of perches in *length* could make up an acre which consists of *surface.*

176. In measuring surfaces, such as square feet of timber, many arithmeticians have called a twelfth part of a square foot, as Ad or Ab, an *inch,* and a twelfth part of Ad, viz. Aa, a *part:* but since a pupil is taught in "Square Measure" that Aa is an inch, and that 144 such inches make up a square foot; it is clearly absurd to call Ad an *inch,* seeing that 12 such make up a square foot: we shall, therefore, confine the name *inch* to Aa, or any quantity equal to it. But the divisions of

FIG. 4.



AC have been, and may be, called superficial *primes, seconds, thirds*, &c.; where it is to be remembered, that a prime is $\frac{1}{12}$th of a square foot, and each succeeding denomination is $\frac{1}{12}$th of the one preceding it. So also, linear inches are sometimes called primes, and twelfths of an inch, seconds.

The square inch, *Aa*, might also be divided, exactly as we have divided *AC*; and its twelfth part would be called a *third*, and be similar in shape, though not in size, to *Ab* or *Ad*; and the twelfth part of this *third* would be called a *fourth*, and be similar in shape to *Aa*.

Also, observing Fig. 4, we learn that

1 foot × 1 foot = 1 sq. foot, as *AC*.

1 foot × 1 inch = 1 sup. prime, as *Ab*.

1 inch × 1 inch = 1 sq. inch, or sup. second, as *Aa*.

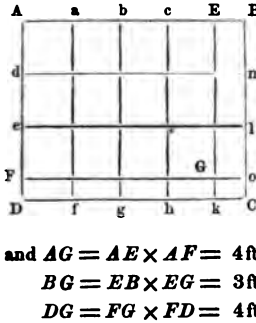
177. It must be distinctly noticed that we do not multiply together concrete quantities, as feet by feet, inches by inches, &c.; we merely multiply the *number* of feet or inches in the length by the *number* in the breadth; and then we observe that the number obtained in the product is equal to the number of superficial units, primes, &c., in the area contained by the above length and breadth. The same remark applies to the multiplication of three dimensions, in finding the volume of a solid.

Attention to this fact, that concrete quantities cannot be multiplied together, will save persons from the absurdity of attempting to multiply pounds, shillings, or pence, by pounds, shillings, &c. I can multiply 5s. by the number 5, and the product is 25s.; but if I multiply 5s. by 5s., I know of no quantity which can correspond to such a product. And in considering (73) it must be noticed, that though in the practice of Rule of Three I appear to be multiplying the third term by the second, which is generally

a concrete quantity, and then dividing by the first term, which is also generally a concrete quantity; yet, since the concrete multiplier and divisor are of the same kind, the result is that I have merely multiplied by a fraction, *i. e.* by an abstract number, generally fractional; and the common process has been, that I have multiplied the third term by the numerator, and divided by the denominator of this fraction.

178. We can now find the area of any rectangular figure contained by commensurable lines.

FIG. 5.



Let $ADCB$ be a rectangular figure, of which the side $AB = 4$ ft. 6 in., and the adjacent side $AD = 3$ ft. 3 in. Let AB be divided into feet in the points a, b, c, E, and AD so divided in the points d, e, F. Draw af, bg, ch, Ek, parallel to AD or BC: and draw dm, el, Fo parallel to AB or DC, according to (175). Then the whole area $AC = AG + BG + DG + CG$.

		sq. ft.	sup. pr.	sq. in.
and $AG = AE \times AF =$	4 ft. $\times$ 3 ft. $=$ 12 sq. ft.	$= 12$	0	0
$BG = EB \times EG =$	3 ft. $\times$ 6 in. $=$ 18 sup. pr.	$= 1$	6	0
$DG = FG \times FD =$	4 ft. $\times$ 3 in. $=$ 12 sup. pr.	$= 1$	0	0
$CG = Go \times Gk =$	6 in. $\times$ 3 in. $=$ 18 sq. in.	$= 0$	1	6
Area AC		$= 14$	7	6

179. Observing these four results, we may place the work in a condensed form, as annexed:

III.	II.	I.
ft.	in.	
4	6	
3	3	
1	1	6
13	6	
14	7	6

Since our proposal was to find the product of 4 ft. 6 in. by 3 ft. 3 in., we place one number under the other nearly as in Compound Multiplication, and commence multiplying the upper line by the lowest denomination in the lower line, and so on through the multiplier. I give the operations which are required to perform the work mentally, observing that each product as it is formed can be reduced to the next higher denomination by dividing it by 12. I commence at the right-hand and proceed thus: $3 \times 6 = 18$, which, divided by 12, gives 1 to carry to column (II.), and 6 to put down: $3 \times 4 = 12$,

for column (II.), which, with the 1 carried, is 13, and divided by 12 gives 1 to carry to (III.), and 1 to set down in (II.). Again, commencing with the multiplier 3 feet, I have $3 \times 6 = 18$ in (II.), which divided by 12 gives 1 to carry to (III.), and a remainder 6 for (II.): lastly, $3 \times 4 = 12$ for (III.), which, with the 1 to carry, becomes 13. Adding the two rows, I have the result 14 sq. ft. 7 sup. primes, 6 sq. in.; or, bringing the primes to square inches, I have 14 sq. feet 90 sq. inches.

Comparing the operations of this article with those of the last, I notice that the steps which produced the first product of the multiplication sum in (179) are the same as the third and fourth in (178); and those which produced the second product are the same as the first and second in that article. Hence in Exs. similar to the one now worked, we need not draw any figure to insure the correctness of the work obtained by the multiplication in (179). This mode of working is called *Cross Multiplication*, and sometimes *Duodecimals*. The latter name is given in consequence of the work of such an Ex. being precisely the same as in Simple Multiplication; provided that, in working from right to left, we take every figure as having *twelve* times the value which it would have one place to the right, instead of ten times that value, as in common numbers.

The above Ex. may be very briefly exhibited in a fractional form.

Thus, the area $= (4\text{ft. }6\text{in.}) \times (3\text{ft. }3\text{in.})$

$= (4\frac{1}{2}) \times (3\frac{1}{4})$ sq. feet,

$= \frac{9}{2} \times \frac{13}{4}$ sq. ft. $= \frac{117}{8}$ sq. feet,

$= 14\frac{5}{8}$ sq. ft. $= 14$ sq. ft. 90 sq. inches, as before.

180. Next, let one or both of the adjacent sides of any rectangular figure, whose area is required, consist of feet, inches, and twelfths of an inch; we must then further divide a square inch as we divided a square foot in (176): *and we then learn* that

1 inch $\times$ $\frac{1}{12}$ inch = $\frac{1}{12}$th of a sq. inch, or 1 super. third;
$\frac{1}{12}$ inch $\times$ $\frac{1}{12}$ inch = $\frac{1}{144}$ super. inch, or 1 superficial fourth.

We can now, without further explanation, follow the work of the accompanying Ex.

Find the area of a rectangular floor, whereof the length is 9 feet 4 inches 7 seconds, and the breadth 5 feet 6 inches 4 seconds.

ft.	pr.	sec.		
9	4	7		
5	6	4		
3	1	6	4	
4	8	3	6	
46	10	11		
51	10	4	0	4

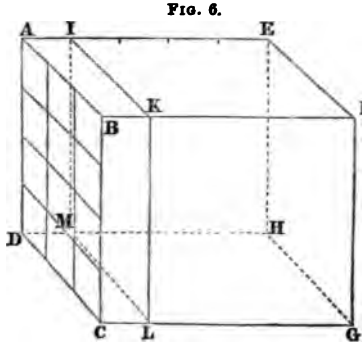
The highest denomination, 51, in this product has principally been obtained from the multiplication of 5 feet and 9 feet, and is therefore square feet; and the remaining denominations are superficial primes, seconds, thirds, fourths: and the whole answer is written

51 sq. feet 10 sup. pr. 4 sec. 0 thirds 4 fourths.

181. We have stated that the length, breadth, and depth of a solid must be multiplied together to obtain its volume; we now proceed to give an Ex. of the truth of this, just as we have already worked one in a question of two dimensions.

First, let all the dimensions consist of whole numbers, viz. 3, 4, 5 inches; and let $ABCGH$ be the solid where $AE = 5$, $AB = 3$, $AD = 4$.

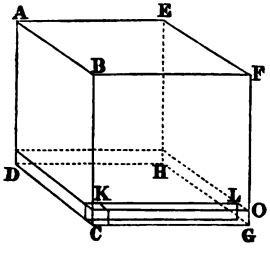
FIG. 6.



Let $IKLM$ be a section of the whole solid by a plane parallel to $ABCD$: then, since $AE = 5$, the whole solid may be divided by similar planes into solid blocks, the base of each of which is $ABCD$, and the thickness 1 inch: hence there will be five times as many solid inches in the whole AG, as there are solid inches in AL.

Now, by (175) the area $ABCD = 4$ in. $\times$ 3 in. $= 12$ sq. in.: and if upon each of these square inches a solid inch be placed, there will be twelve solid inches in AL. Hence, in the whole AG, *i. e.* in five times AL, there will be 5×12 cubic inches; *i. e.* the volume of the solid $= 5 \times 4 \times 3$ solid inches. Of course, if the three dimensions had been expressed in feet instead of inches, the result would have been $5 \times 4 \times 3$ solid feet.

FIG. 7.



Let now the solid have all its dimensions equal, and each = 1 foot, then the figure is a cubic foot, and its volume = 12 in. × 12 in. × 12 in. = 1728 cubic inches, and 12×12×12, or 12^3 is called 12 cubed. Hence we can obtain the numbers exhibited in what is called "Solid Measure."

For 12in. × 12in. ×12in., or 1728 solid inches = 1 solid foot;

3ft. ×3ft. ×3ft., or 27 solid feet = 1 solid yard.

By a classification similar to that of (176), we term a twelfth part of a solid foot, a solid or cubic prime; a twelfth of a prime, a second; and so on: but we must always remember, that though we use but one set of names, we have three kinds of primes, seconds, &c. viz. linear, superficial, and solid; and the various sorts can never be added to or subtracted from each other—for it is evidently impossible to add an area either to a line or a volume.

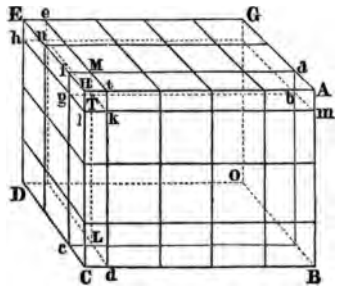
By observing Fig. 7 we learn that

1 sq. foot × 1 linear inch = 1 solid prime, as *DO*.

1 sup. prime × 1 linear inch = 1 solid second, as *CL*.

1 sq. inch or 1 sup. sec. × 1 linear inch = 1 solid third, or solid inch, as *CK*.

FIG. 8.



Let *ABCDEG* or *BE* be a solid contained by rectangular surfaces; let the three adjacent edges, *CB, CD, CH*, be respectively equal to 4ft. 5in., 2ft. 6in., and 3ft. 4in.; let the surfaces *AC, CE, EA*, be intersected by parallel lines drawn through the extremities of the feet in *their respective* sides, as in (178); let *M* be the point in

the surface AE, in which the lines aMf, tMe, bounding the complete squares in AE, intersect; ml, lh, the lines bounding the squares in the surfaces CE, AC, intersect in l, a point in the edge CH.

Through aMf draw a plane ac parallel to AC

„ eMt „ ed „ EC

„ mkl „ hm „ AE

The planes ac and ed shall intersect in the line MTL

„ ac „ hm „ gTb

„ ed „ hm „ kTn

These three lines all pass through the point T; and, measuring from T, the figure may be divided into the following solids:—

(1) TO, TB, TC, TD, upon the bases LO, LB, LC, LD, and all having the same altitude LT.

(2) TG, TA, TH, TE, upon the bases MG, MA, MH, ME, (which are precisely the same as the bases of the other four,) and having an altitude MT.

We shall now find the volume of each of these solids according to (181); and show that a process in Multiplication, similar to that in (179), will produce a result corresponding to the value actually obtained from the figure.

182. By (178) we may at once write, that

The area of $GH = EG \times GA =$ (4 ft. 5 in.) × (2 ft. 6 in.)
= 11 sq. ft. 0 pr. 6 sec. (I.)

and from (174) we ought to have the volume of GC,
= (11 sq. ft. 0 pr. 6 sec.) × (3 ft. 4 in.) (II.)

By observing the four products in the multiplication of (I.), we find that they are respectively

sq. ft.	pr.	sec.	
8	0	0	= area GM or OL
2	0	0	= „ AM or BL
	10	0	= „ EM or DL
		30	= „ HM or CL
11	0	6	= area GH or DB

Hence we may write the product (11.) in the following shape:

sq. ft. pr. sec.						sol. ft. pr. sec. thir.	sol. ft. pr. sec. thir.	
8 0 0 3 ft. 4 in.	$= \left\{\begin{matrix} OL \times TL \\ GM \times TM \end{matrix}\right\} = \left\{\begin{matrix} OT \\ GT \end{matrix}\right\} = LG =$					$\left\{\begin{matrix} 24\ 0\ 0\ 0 = \\ 32\ 0\ 0 = \end{matrix}\right.$	$\begin{matrix} 24\ 0\ 0\ 0 \\ 2\ 8\ 0\ 0 \end{matrix}$	$\begin{matrix} B \\ A \end{matrix}$
2 0 0 3 ft. 4 in.	$= \left\{\begin{matrix} BL \times TL \\ AM \times TM \end{matrix}\right\} = \left\{\begin{matrix} BT \\ AT \end{matrix}\right\} = LA =$					$\left\{\begin{matrix} 6\ 0\ 0\ 0 = \\ 8\ 0\ 0 = \end{matrix}\right.$	$\begin{matrix} 6\ 0\ 0\ 0 \\ 8\ 0\ 0 \end{matrix}$	$\begin{matrix} B \\ A \end{matrix}$
10 0 3 ft. 4 in.	$= \left\{\begin{matrix} DL \times TL \\ EM \times TM \end{matrix}\right\} = \left\{\begin{matrix} DT \\ ET \end{matrix}\right\} = LE =$					$\left\{\begin{matrix} 30\ 0\ 0 = \\ 40\ 0 = \end{matrix}\right.$	$\begin{matrix} 2\ 6\ 0\ 0 \\ 3\ 4\ 0 \end{matrix}$	$\begin{matrix} B \\ A \end{matrix}$
30 3 ft. 4 in.	$= \left\{\begin{matrix} CL \times TL \\ HM \times TM \end{matrix}\right\} = \left\{\begin{matrix} CT \\ HT \end{matrix}\right\} = LH =$					$\left\{\begin{matrix} 90\ 0 = \\ 120 = \end{matrix}\right.$	$\begin{matrix} 7\ 6\ 0 \\ 10\ 0 \end{matrix}$	$\begin{matrix} B \\ A \end{matrix}$
11 0 6 3 ft. 4 in.	=				BE	=	36 9 8 0	

183. Taking the area GH or DB, and multiplying it by 3 ft. 4 in. by cross multiplication, we have the annexed operation, where it will be found that the former of the two products equals the sum of all the products (A), and the latter equals the sum of all the products (B): and the result is 36 sol. ft. 9 pr. 8 sec.; or 36 sol. ft. 1392 sol. in.

sq. ft. pr. sec.	
11 4 6	
3 ft. 4 in.	
3 8 2 0	= (A) or vol. Gl
33 1 6	= (B) or vol. Ol
36 9 8 0	= vol. GC

The above value of the volume may be written

$$\left(36 + \frac{9}{12} + \frac{8}{144}\right) \text{ feet, or } 36 + \frac{108+8}{144}, \text{ or } 36\tfrac{116}{144}, \text{ or } 36\tfrac{29}{36} \text{ feet;}$$

and this result might, as in (179) have been obtained at once, by expressing the three dimensions as fractional parts of feet, and finding their product:

Thus (4ft. 5in.) × (2ft. 6in.) × (3ft. 4in.) = $4\frac{5}{12} \times 2\frac{1}{2} \times 3\frac{1}{3}$ solid feet,

$$= \frac{53}{12} \times \frac{5}{2} \times \frac{\overset{5}{10}}{3} = \frac{53 \times 25}{36} = \frac{1325}{36} = 36\frac{29}{36} \text{ solid feet.}$$

If the dimensions of the solid be expressed in feet, inches, and twelfths the above method of Cross' Multiplication may be employed, and we shall find the result expressed in a form similar to the last, but with terms descending lower than solid inches, which was the lowest denomination in the last Example.

183. Since length × breadth × depth = volume (1.), therefore length × breadth = $\frac{\text{volume}}{\text{depth}}$; and since length × breadth is of two dimensions, therefore $\frac{\text{volume}}{\text{depth}}$ is of two dimensions, *i.e.* $\frac{3 \text{ dimensions}}{1 \text{ dimension}}$ gives two dimensions.

So also, from (1.) length = $\frac{\text{volume}}{\text{breadth} \times \text{depth}}$; *i. e.* $\frac{3 \text{ dimensions}}{2 \text{ dimensions}}$ gives one dimension: and any fraction in which numerator and denominator are of the same dimension, is an abstract number. (See Art. 66.)

184. The following Exs., though not involving Cross Multiplication, are amongst the most useful of those in which the measurement of surfaces and solids occurs.

Ex. I. Find the number of acres in a rectangular field, of which the length is 35 chains 72 links, and the breadth 24 chains 6 links.

To understand this question, a pupil must know that large pieces of land are measured by means of a chain called *Gunter's Chain*, which is four poles, or 22 yards, in length, and is divided into 100 equal parts, called *Links*.

Also, an acre is equal to a rectangular surface of which the length is 40 poles, or 10 chains, and breadth 4 poles, or 1 chain: hence,

the area of an acre = 10 chains × 1 chain = 1000 links × 100 links = 100,000 square links. Consequently, if the dimensions of a field be expressed in links, and its area thence obtained in square links, this value, when divided by 100,000, will be expressed in acres; *i.e.* if five places be pointed off as a decimal, the result will be acres and decimal parts of an acre, which can be reduced to roods and poles. Returning now to the Ex. we have

Length = 35 chains 72 links = 3572 links
Breadth = 24 chains 8 links = 2408 links

28576
142880
7144
Area = acres 86·01376
4
Roods ·05504
40
Poles 2·20160

and therefore the field contains 86 a. 0 r. 2$\frac{1}{5}$ poles, nearly.

Ex. II. Find how many gallons are contained in a cistern of which the length is 40 inches, breadth 36 inches, and depth 16 inches.

It must here be observed that an imperial gallon is equal to 277·274 cubic inches: hence, when the number of solid inches in any volume is known, the number of gallons which it will contain is found by dividing those solid inches by 277·274. In the above Ex. we therefore have

the solid content = (40 × 36 × 16) solid inches;

and therefore number of gallons contained = $\frac{23040}{277{\cdot}274}$

Ex. III. A roof of 27$\frac{1}{2}$ feet by 18$\frac{3}{4}$ is to be covered with lead weighing 8lbs. per square foot: what would it cost at the rate of £5 4s. for 5cwt.?

The area of the roof = 27$\frac{1}{2}$ ft. × 18$\frac{3}{4}$ ft.

$$= \left(\frac{55}{2} \times \frac{75}{4}\right) \text{sq. ft.}$$

$$= \frac{4125}{8} \text{ sq. ft.}$$

hence, weight of lead = $\frac{4125}{8}$ × 8lbs. = 4125lbs.

To find the cost of the lead, we have the following statement:

5 × 112lbs. : 4125lbs. :: £5 4s.

The fourth term will be found to be £38 6s. 0$\frac{8}{11}$d.

EXTRACTION OF ROOTS.

185. We have already seen (92) that a number, by being successively multiplied by itself, is said to be raised to a power; and the order of the power, whether it be second, third, fourth, &c., depends upon the number of times the original number is to be repeated. This process is termed *Involution;* and the reverse process of obtaining the original number from the power is called *Evolution.* The original number is called, with respect to its power, the *Root;* and this evolution is also termed *Extraction of Roots.*

Thus, if 5×5, or 5^2, or 5 squared (175), as it is called, $= 25$; 5 is called the Square Root of 25. Similarly, since $6 \times 6 \times 6$, or 6^3, or 6 cubed (181) $= 216$, then 216 is termed the cube or third power of 6, and 6 the cube root or third root of 216.

The sign $\surd$ is used to express the operation of extracting a root; and a small figure, placed thus $\sqrt[3]{}$, shows what root [here the *third* root] is to be extracted: but the figure is generally omitted when the sign refers to the square root; and the sign $\surd$ itself indicates extraction of the Square Root.

186. Any number or quantity which is thus formed by the squaring, cubing, &c. of any number, is called a complete or perfect square, cube, &c.; and therefore the square, cube, &c. root, of such a power can be exactly extracted: but any number, as 20, which lies between two complete squares, 16 and 25, cannot have its root obtained exactly. But since 16 has a root 4, and 25 has a root 5, therefore the root of 20 will be between 4 and 5, or its value will be approximately expressed by the mixed decimal 4·472...

and similarly of every number which lies between two complete squares. So also every number which lies between two complete cubes must have its cube root expressed in a decimal form.

187. If a number be raised to a power, and then the same root of that power be extracted, we shall obtain the original quantity; therefore $\sqrt{3^2} = 3$; $\sqrt[3]{5^3} = 5$.

Also, since the square root of a given number may be defined to be that number which, when multiplied by itself, will produce the given number, therefore $\sqrt{2} \times \sqrt{2} = 2$: so also, $\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2} = 2$. Such quantities as $\sqrt{2}$, $\sqrt[3]{2}$, are called *Surds*, or *Irrational* quantities.

Obs. Since fourth, fifth, &c. powers of numbers can be obtained, of course fourth, fifth, &c. roots must also exist: but we shall here confine ourselves to the finding of square and cube roots.

What has been said of the raising whole numbers to powers is also true of fractional quantities, whether vulgar, or commensurable decimals. Thus $\frac{2}{3} \times \frac{2}{3}$, or $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$; therefore $\sqrt{\frac{4}{9}} = \frac{2}{3} = \frac{\sqrt{4}}{\sqrt{9}}$; *i. e.* the square root of a fr^{n} must be obtained by taking the square root of numerator and denominator. In like manner, since $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$, or $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$; therefore $\sqrt[3]{\frac{8}{27}} = \frac{2}{3} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}}$; *i. e.* the cube root of a fraction will be found by taking the cube root of both numerator and denominator. If the root of a mixed number be required, it must be reduced to an improper fraction or a decimal. Thus:

$$\sqrt{1\tfrac{9}{16}} = \sqrt{\tfrac{25}{16}} = \frac{\sqrt{25}}{\sqrt{16}} = \frac{5}{4} = 1\tfrac{1}{4}.$$

We shall presently show how to extract the square root of *decimals* as well as of whole numbers.

SQUARE ROOT.

188. Before proceeding to investigate a rule for the extraction of the Square Root, we must remember the following:

Digits	1,	2,	3,	4,	5,	6,	7,	8, 9.
Squares	1,	4,	9,	16,	25,	36,	49,	64, 81.

From what has been said, it is plain that the square of 1, is 1; of 10, is 100; of 100, is 10,000, &c. &c.

Therefore the square root of		1	is	1	
„	„	100	„	10	
„	„	10,000	„	100	
„	„	1,000,000	„	1,000	&c. &c.

Hence, if a number lie between 1 and 100, *i. e.* have not more than two figures, its square root is between 1 and 10, or has not more than one figure.

If between 100 and 10,000, *i. e.* have not more than four figures, the root is between 10 and 100, or has not more than two figures.

If between 10,000 and 1,000,000, *i. e.* have not more than six figures, the square root is between 100 and 1,000, or has not more than three figures: and so on.

So that if over every alternate figure of any number, beginning at the units' place, a point be placed, the number of points will show the number of figures in the square root. For example, 3757 has more than two figures, and not more than four, or lies between 100 and 10,000, and therefore its root lies between 10 and 100, *i. e.* has not more than two figures, and the number must therefore be thus pointed, $3\dot{7}5\dot{7}$; so also $1\dot{3}4\dot{5}9$ is correctly pointed, for there will be three figures in the square root. The divisions which are formed by these points are called *periods*. The periods in $1\dot{3}4\dot{5}9$ are 1, 34, and 59.

Also, since the square of ·1 is ·01, therefore

the square root of	·01	is	·1	
So also, of	·0001	is	·01	
„ „	·000,001	is	·001	&c. &c.

It hence appears that in the square of any decimal an even number of decimal places will always be found; and that if there be an odd number in any proposed Ex., the number of places must be made even, by appending a cipher, which cannot alter the value of the decimal. We then

point every alternate figure of a decimal, from left to right, beginning with the second figure from the units' place, so that the last figure will always be pointed.

189. The Rule for the Extraction of the Square Root is derived from an algebraical operation, wherein a complete square is taken, and a process is then contrived by which the root, which is already known, can be deduced from the complete square.

I shall therefore proceed, contrary to my usual method, to give a Rule for extracting the Square Root, without any more explanation than is necessary for the mere working of Exs.: and the arithmetical illustration of the above algebraical process, which I shall afterwards give, may be read or omitted at the reader's pleasure.

RULE. Divide the given number into periods. Find the greatest square number which is not greater than the first period; subtract it from that period, and place the root of the number in the quotient.

To the remainder, after subtraction, bring down two figures or one period, as in Long Division, and consider the whole as a dividend.

For a divisor, double the quotient, and try how often it is contained in the dividend, except the last figure: the figure thus obtained by division place in the quotient, and annex it to the divisor.

Multiply this whole divisor by the last figure in the quotient, and subtract this *subtrahend* from the dividend.

Bring down another period; find a fresh divisor by adding the last figure of the former divisor to that divisor, and proceed exactly as before.

OBS. It will be found that on dividing by the *incomplete* divisor—especially when the early figures in the quotient

are small, and the latter ones large—there will result a quotient larger than the one which must be taken. The reason of this will be explained hereafter.

I will work one Ex. which gives no remainder, *i. e.* where the given number is a perfect square, viz. 81796.

```
 8̇17̇96̇ (286
   4
 48) 417
     384
566)  3396
      3396
     ====
```

The greatest square number not greater than the first period, 8, is 4; I therefore subtract 4 from the 8, and place in the quotient, 2, the root of the 4: to the remainder, 4, I bring down the second period, 17, and consider 417 as my dividend. For a divisor, I double the 2 in the quotient: and on dividing 41 by this divisor 4, I obtain a quotient 10; but upon trial I find that 8 is the largest quotient that can be employed. I place the 8 in the quotient, and at the right of the 4 in the divisor, making a *complete* divisor 48: multiplying this 48 by the 8, I have a subtrahend 384; the remainder, after subtraction, is 33, and with the third period 96 gives a new dividend 3396: adding the last figure 8 in the divisor 48, to that 48, I have, as new partial divisor, 56, which goes six times in 339: placing the 6 in the quotient and in the divisor, and multiplying the whole divisor 566 by the 6, I have a subtrahend 3396, which, upon subtraction, leaves no remainder. The square root is therefore 286.

Since most of the numbers we meet with are not perfect squares, we can obtain only approximate roots of such numbers; and the operation is carried to about three places of decimals in the root, requiring, of course, six places in the number. If the given number be an integer, or have not so many as six decimal places, the number must be made up by appending ciphers.

Ex. II. Find the square root of 876·535.

```
 8̇76·53̇50̇0̇0̇ (29·606..
   4
 49) 476
     441
586)  3553
      3516
59206)  37500
        355236
         19764
```

I append three ciphers, and point according to (188). When the third divisor, 592, is obtained, the quotient is 0; I therefore place the 0, as usual, in the quotient and in the divisor, then bring down another period, and proceed as before. Since there were two periods in the integral part of the given number, there will be two integers in the root, and the decimal point must be placed after the 29.

190. I will now illustrate the algebraical process mentioned in (189), as far as can be done in arithmetic.

Let the square number 169 be taken, the root of which is 13. If this number and its root be expressed in the required algebraic form, they will be written $100 + 60 + 9$, and $10 + 3$.

$$\begin{array}{rll} & 100+60+9 & (10+3 \\ & 100 & \\ \hline 20+3) & 60+9 & \\ & 60+9 & \\ \hline\hline \end{array}$$

Placing in the quotient the $10 + 3$, which is known to be the root, I observe that the first part of the root, viz. 10, is the square root of 100, the first part of the number: I may therefore consider that the square root of the 100 gives the 10, and the remainder $60 + 9$ is to furnish the 3. Observing the former part of the remainder, viz. 60, I notice, that if it were divided by twice the 10 in the quotient, I should obtain the required number, 3: I therefore make 20, *i. e.* twice the quotient, my divisor. Also, since there must be no remainder, I must have as subtrahend $60 + 9$; and since the 9 is the square of 3, therefore, if I append the 3 to the 20 in the divisor by the sign (+), and multiply the whole divisor $20 + 3$ by 3, I shall obtain a subtrahend equal to the dividend $60 + 9$, and therefore shall have no remainder, as was required.

An algebraical process similar to the above would *prove* that this method of procuring a divisor and subtrahend would *always* succeed in bringing no remainder, *i. e.* in obtaining the exact root of a complete square. But this Ex. of course only *illustrates* the method of proof, and shows how the Rule is algebraically deduced. The operation, when condensed into an arithmetical form, stands thus:

$$\begin{array}{rl} & \dot{1}6\dot{9}\ (13 \\ & 1 \\ \hline 23) & 69 \\ & 69 \\ \hline\hline \end{array}$$

191. I will work one more Ex. which will show how an error in the quotient may arise in dividing by an incomplete divisor.

$(27)^2 = 729$, which, algebraically represented, is $400 + 280 + 49$: and the quotient or root, is $20 + 7$.

$$\begin{array}{r|l} 400 + 280 + 49 & (20 + 7 \\ 400 & \\ \hline 40 + 7) & 280 + 49 \\ & 280 + 49 \\ & \hline\hline \end{array}$$

And this, when condensed, is

$$\begin{array}{r|l} \dot{7}2\dot{9} & (27 \\ 4 & \\ \hline 47) & 329 \\ & 329 \\ & \hline\hline \end{array}$$

Obs. Since to divide 280 or 320 by 40 is the same as to divide 28 or 32 by 4; I may therefore call 4 the divisor, when 28 or 32 is the dividend, and consider 40 as the divisor, when 280 or 320 is the dividend.

If this Ex. be worked in the usual manner, it will be found that the incomplete divisor 4 would go eight times in 32, whereas the real quotient is found to be only 7. By observing the algebraic method, I learn that when the dividend 329 is separated into its parts $280 + 49$, the true quotient is obtained by dividing only the former part, 280, by the 40, or 28 by 4: but in the arithmetical operation I cannot see how much of 329 is the former part which will give me a correct quotient; and therefore I have to divide the whole 329 by 40, or 32 by 4, and so run the risk of an error. If the second part of the dividend 49 had been less than 40, the quotient would at once have appeared to be 7, and no error arisen. Hence, the larger the second part of the dividend (as 49) is, when compared with the incomplete divisor (as 40), the greater will be the error: and since the 4 in the divisor was obtained from the *former* part of the quotient, viz. 2, and 49 from the latter part, viz. 7, the error will be the greatest when the earlier figures of the quotient are small, and the latter are large.

This method of proof is not limited to numbers, the roots of which consist but of two figures: but any attempt to

extend the illustration would be very cumbrous without the use of algebra.

192. We have seen that the area of a square is obtained by squaring any one of its sides; hence, the number in the side of a square is found by extracting the square root of the number which represents an area.

The number which represents the area of a square figure may not always be a complete square: for instance, if in Fig. 1, p. 134, a square be described with sides equal to ac or $\sqrt{2}$, then its area would be $(\sqrt{2})^2 = 2$, which is not a perfect square. When, therefore, the number representing the area of a square is not itself a perfect square, the number representing the side will be incommensurable or irrational.

In obtaining the square root of a fractional quantity, we may extract the root of numerator and denominator, if the denominator be a complete square: but if not, it is better to reduce the fraction to a decimal, and then extract the root. For example:

$$\sqrt{27\tfrac{3}{128}} = \sqrt{\frac{3483}{128}} = \frac{59{\cdot}01\ldots..}{11{\cdot}31\ldots.}$$

and to obtain the result I should have to extract two roots, and perform an operation in Long Division; whereas, since $27\frac{3}{128} = 27{\cdot}2109375$, a single extraction will suffice to give the answer at once.

In finding the root of a circulating decimal, it will sometimes happen that the equivalent vulgar fraction will have both numerator and denominator complete squares, and we can then readily extract the root. Thus:

$$\sqrt{1{\cdot}\dot{7}} = \sqrt{1\frac{7}{9}} = \sqrt{\frac{16}{9}} = \frac{4}{3} = 1\tfrac{1}{3}, \text{ or } 1{\cdot}\dot{3}.$$

But as this is rarely the case, we must generally find the approximate root by the usual method, though it will not be a recurring decimal.

193. Sometimes a fraction involving a surd may be in its lowest terms, but yet not in a form the most convenient for finding its value. Thus $\frac{1}{\sqrt{2}}$ is in the lowest terms; but multiplying numerator and denominator by $\sqrt{2}$, it becomes $\frac{\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}$, or $\frac{1}{2}\sqrt{2}$.

Also, since $\sqrt{2} = 1{\cdot}4142\ldots$ therefore $\frac{1}{\sqrt{2}} = \frac{1}{1{\cdot}4142\ldots}$ which will require a Long Division sum to bring out its value; but $\frac{1}{2}\sqrt{2} = \frac{1}{2}(1{\cdot}4142\ldots) = {\cdot}7071\ldots\ldots$ by mere Short division. Hence $\frac{1}{2}\sqrt{2}$ is more simple than $\frac{1}{\sqrt{2}}$.

And as a general rule, all quantities involving surds are in their simplest form, when the surds are in the numerator and not in the denominator.

The following Exs. illustrate the operations of the last few pages.

Ex. I. A square field contains 15a. 2r. 20p. Find its side in chains.

15a. 2r. 20p. = 2500 sq. poles; therefore the side of a field containing 2500 sq. poles = 50 linear poles

$= \frac{50}{4}$ linear chains (since 4 poles = 1 chain.)

$= 12\frac{1}{2}$ linear chains.

Ex. II. Two acres of land are to be cut from a rectangular field of which the breadth is 20 chains 50 links, by a line parallel to it. Find the length of the plot.

1 acre = 40p. × 4 poles = 10 chains × 1 chain = 10 sq. chains; and length × breadth = 2 acres; or, since the breadth is $2\frac{1}{2}$ chains, therefore length × $2\frac{1}{2}$ chains = 2 × 10 sq. chains = 20 sq. chains.

$$\text{Hence length} = \frac{20 \text{ sq. chains}}{2\frac{1}{2} \text{ linear chains}}$$

$$= \overset{4}{\cancel{20}} \times \frac{2}{\cancel{5}} \text{ linear chains (183)}$$

$$= 8 \text{ linear chains.}$$

CUBE ROOT.

194. Since the subject of Square Root has been fully discussed, there will be the less need for dwelling at great length upon that of Cube Root.

The following numbers must be remembered:

Digits	1,	2,	3,	4,	5,	6,	7,	8, 9,
Cubes	1,	8,	27,	64,	125,	216,	343,	512, 729:

and it will be observed that no two of these cubes end with the same digit: hence, if a given number be a perfect cube, the last figure in its root may be known. For instance, if a cube number end in 3, its cube root ends in 7: if the number end in 2, its root ends in 8. Recollection of this fact is often of service.

According to the method of (188), we learn that the cube root of

1 is 1
1,000 „ 10
1,000,000 „ 100 &c. &c.

hence, if a number between 1 and 1000, *i.e.* have not more than three figures, its cube root is between 1 and 10, or has not more than one figure.

If the number be between 1000 and 1,000,000, *i. e.* have not more than six figures, its root has not more than two figures: and if the number have not more than nine figures, its root has not more than three figures.

So that if over every third figure, beginning at the units' place, a point be placed, the number of points will show the number of figures in the cube root.

Similarly, since the cube of	·1	is ·001
therefore the cube root of	·001	„ ·1
So also,.....	·000,001	„ ·01
„ „ „	·000,000,001	„ 001 &c. &c.

It hence appears that the number of places in the cube of any decimal must always be some multiple of 3: and if the number of places in any

decimal, of which we have to find the cube root, be not a multiple of 3, it must be made so by appending ciphers: we therefore commence pointing at the units' place, and point every third figure to the left over the integers; and to the right over the decimals, if there be any.

195. I will now give the Rule for the extraction of the Cube Root: but as an arithmetical illustration of the algebraical process, similar to the one given in Square Root, would be very cumbrous, I prefer giving the algebraical proof itself in a Note, and leave it to be read only by those who have mastered the Elementary Rules in Algebra.

RULE. Divide the given number into periods. Find the greatest cube number which is not greater than the first period. Subtract it from that period; place the root of the number in the quotient, and form a dividend, as in Square Root. For a divisor, multiply the square of the quotient by 3, and try how often it is contained in the dividend, except the last two figures; the figure thus obtained by division place in the quotient.

To form the subtrahend: consider the quotient made up of two parts, whereof one is the last figure alone, and the other is the former figure or figures with a cipher annexed. Cube this last figure, and to it add three times the product of the two parts of the quotient by the whole quotient.

Subtract this subtrahend from the dividend; to the remainder, if any, bring down another period,—find a fresh divisor by again multiplying the square of the quotient by 3, and proceed precisely as before.*

* Let $(a+b)^3$, or $a^3+3a^2b+3ab^2+b^3$ be a perfect cube, from which I am to deduce a rule for the Extraction of the Cube Root. Of course the required root is $a+b$.

$$\begin{array}{ll} a^3+3a^2b+3ab^2+b^3 & (a+b \\ a^3 & \\ \hline 3a^2) \quad 3a^2b+3ab^2+b^3 & \\ \qquad\qquad\qquad\qquad b^3=b^3 & \\ \hline 3a^2b+3ab^2 \qquad\qquad = 3ab\,(a+b) & \end{array}$$

Obs. As in Square Root, the divisor often gives upon trial too large a quotient.

Ex. Find the cube root of 185193.

$$
\begin{array}{rl}
 & 185\dot{1}9\dot{3}\ (57 \\
 & 125 \\
\hline
3\times 5^2 = 75) & 60193 \\
 & 343 = 7^3 \\
\hline
 & 59850 = 3\times 50\times 7\times 57 \\
\hline
 & 60193 \\
\hline\hline
\end{array}
$$

Placing a point over the 3 and the 5, I find that 5^3, or 125, is the greatest cube number below 185. I therefore subtract 125 from 186, place 5 in the quotient, and, bringing down the next period, have 60193 as dividend. The divisor $= 3\times 5^2 = 75$, which appears to go eight times, but the real quotient is 7. According to the Rule, I now consider the quotient 57 as composed of two parts, 50 and 7: and for a subtrahend I take first 7^3, or 343: next, I have $3\times 50\times 7\times 57 = 59850$; and the sum of these results forms a subtrahend 60193, which, when subtracted from the dividend, leaves no remainder. Hence 57 is the exact cube root of 185193.

196. The following Ex. involves decimals, and gives only an approximate root.

$$
\begin{array}{rl}
 & 2\dot{1}0\dot{3}5\cdot 8\dot{0}\dot{0}\ (27\cdot 6 \\
 & 8 \\
\hline
3\times 2^2 = 12) & 13035 \\
 & 343 = 7^3 \\
\hline
 & 11340 = 3\times 20\times 7\times 27 \\
\hline
 & 11683 \\
\hline
3\times (27)^2 = 2187) & 1352800 \\
 & 216 = 6^3 \\
\hline
 & 1341360 = 3\times 270\times 6\times 276 \\
\hline
 & 1341576 \\
\hline
 & 11224 \\
\hline\hline
\end{array}
$$

Here it is plain that the first portion of the root is found by taking the cube root of a^3, the first portion of the given quantity. Also b, the second part of the root, is obtained by dividing $3a^2b$, the first portion of the dividend, by $3a^2$, or three times the square of the a already obtained in the root.

The subtrahend is formed of two parts, b^3 and $3a^2b + 3ab^2$: this latter portion $= 3ab\,(a+b)$: and these two algebraical expressions, when enunciated in words, give the process for forming the subtrahend mentioned in the Rule.

Since in Arithmetical Examples the dividend is expressed as one quantity, but the *true* quotient b is found by dividing only a *part* of that dividend; therefore if the latter *portion of the dividend*, viz. $3ab^2 + b^3$, be large in proportion to the former part,

I here observe that in forming the second subtrahend, I consider the quotient as made up of 270 and 6. By bringing down more periods of three ciphers, I might carry the root to several places of decimals; but the work becomes exceedingly heavy when the quotient extends beyond three figures. Since one period consisted of decimals, the last figure in the quotient must be pointed off as a decimal.

The observations which were made upon the extraction of the Square Root of Fractions in (192) apply also to a similar use of the Cube Root.

Just as the number in the side of an area of square form is found by extracting the square root of the number representing the area: so the number in the edge of a volume of cubic form is found by extracting the cube root of the number which represents the volume.

I will work one or two Exs. to illustrate the application of Cube Root.

Ex. III. A beam is 6ft. 9in. long, 2ft. 3in. broad, and 9in. thick; required the side of a cube of equal capacity.

Working fractionally we have

Solid content of beam $= 6\frac{3}{4}$ ft. $\times\ 2\frac{1}{4}$ ft. $\times\ \frac{3}{4}$ ft.

$$= \left(\frac{27}{4} \times \frac{9}{4} \times \frac{3}{4}\right) \text{ sol. ft.} = \frac{729}{64} \text{ sol. ft.}$$

$$\text{or} = \frac{9^3}{4^3} \text{ sol. feet.}$$

$$\text{therefore the edge of cube} = \sqrt[3]{\frac{9^3}{4^3}} \text{ sol. ft.} = \frac{9}{4} \text{ lin. feet} = 2\tfrac{1}{4} \text{ linear feet.}$$

$3a^2b$, then the quotient obtained by division will be too large; and the error will be greatest when b is large and a small. If there be more than two terms in the quotient, b must always represent the last term alone, and a the whole of the other terms.

Also, since in division in Algebra the first term of the divisor is alone useful in finding the quotient, therefore (in the above method of working Algebraical Examples,) since $3a^2$ will always be the first term of the divisor, however many terms may be in the quotient, the first divisor, $3a^2$, will serve for every dividend throughout the Example.

Ex. IV. Find the area of any one of the six surfaces of a cube containing 11 cubic feet, 675 cubic inches.

To find the area of this surface, I must find the edge of the cube, and square it. Reducing the cubic feet to inches, I have

$$\text{volume of cube} = 19683 \text{ cubic inches};$$

$$\text{therefore edge of the cube} = \sqrt[3]{19683} = 27 \text{ linear inches} = 2\text{ft. } 3\text{in};$$

$$\text{and therefore area of the side} = (2\tfrac{1}{4})^2 = \left(\frac{9}{4}\right)^2 = \frac{81}{16} \text{ sq. ft.}$$

$$= 5\tfrac{1}{16} \text{ sq. ft.}$$

$$= 5\,\text{sq. ft. } 0\,\text{pr. } 9\,\text{sq. in.}$$

Ex. V. If the volume of any cylinder equals its length × area of its base, find the value of this area in a cylindrical wire 50 feet long, and made out of a square inch plate of metal ·05 inches in thickness.

Here, since area × length = whole volume of metal

$$= 1 \text{ sq. inch} \times \text{thickness of the plate};$$

$$\text{or, since Area} \times 50 \text{ ft.} = 1 \text{ sq. inch} \times \cdot 05 \text{ inches};$$

$$\text{therefore Area} = \frac{1 \text{ sq. in.} \times \cdot 05 \text{ in.}}{50 \times 12 \text{ in.}} \qquad \text{(G)}$$

$$= \frac{\cdot 05}{50 \times 12} \text{ sq. inches}$$

Multiplying numerator and denominator by 100 ..

$$= \frac{5}{5000 \times 12} \text{ sq. inches}$$

$$= \frac{1}{12000} \text{ sq. inches}$$

The right-hand side of (G) gives $\frac{3 \text{ dimensions}}{1 \text{ dimension}}$; which, according to (183), is 2 dimensions, as an Area ought to be.

Obs. An expression which is of the same dimensions throughout, as (G), is said to be *homogeneous*.

APPENDIX.

FRACTIONAL QUOTIENT.

197. It was mentioned in (22) that the word quotient is not inapplicable, where one number is to be divided by another, and yet the division cannot be performed; thus, in the division of 2 by 5, we said that the quotient was $\frac{2}{5}$.

This may be seen more clearly, if we remember that the word *quotient* means *how often* the divisor is contained in the dividend,—and if we observe the following illustration. Suppose that I have a rod five inches long, and with it I am to measure the depth of water in several full vessels, of which the depth is known to be 1, 2, 3, 4 inches. If now I ask *how many* times the length of this rod is contained in the depth of the water; the reply is, that in the first, one inch or one-fifth of the rod was immersed, *i. e.* the rod *went into* the depth one-fifth times; and therefore the genuine quotient of 1 when divided by 5 is one-fifth or $\frac{1}{5}$. Similarly, the quotient in the case of the second, third, &c. vessels, would be $\frac{2}{5}$, $\frac{3}{5}$, &c. And when I come to measure the depth of the five inch vessel, the quotient ought to be $\frac{5}{5}$, or 1, *i. e.* the rod is just once immersed. And when I measure the seven inch depth, I find the quotient to be $\frac{7}{5}$ or $1\frac{2}{5}$; *i. e.* it takes one immersion of the rod, and $\frac{2}{5}$ of another, to measure the seven inches.

COMPOSITE DIVISOR.

198. In the operations of Reduction we frequently have to divide by composite divisors, which consist of the product of two factors, each not exceeding 12; and by dividing by these component parts, we can employ Short instead of Long Division. For instance, in reducing grs. to dwts., I have to divide by 24: but since $24 = 6 \times 4$, it will be seen that if I divide by 6 and 4 successively, I obtain a correct result; but the main difficulty is to obtain the true remainder.

Ex. Convert 3887 grains into dwts.

$$24 \left\{ \begin{array}{r} 6)\ 3887 \\ 4)\ 647 \quad 5 \\ 161 \quad 3 \end{array} \right\} 23 \text{ grs.}$$

Dividing by 6, we have a quotient 647 and remainder 5. Now, dividing the grs. by 6, is equivalent to separating them into parcels, each containing 6 grs., and the remainder is of the same kind as the 3887, viz. grs.; *i. e.* 3887 grs. = 647 parcels of 6 grs. + 5 grs.

Again, since four parcels of 6 grs. each will make one parcel of 24 grs., *i. e.* 1 dwt., therefore, upon dividing the 647 by 4, the quotient will become dwts.; and the remainder, if any, will be parcels of 6 grs. each: the quotient is 161, and remainder 3. Hence the whole remainder = 3 parcels of 6 grs. + 5 grs., or = 3 × 6 grs. + 5 grs. = 18 grs. + 5 grs. = 23 grs. And if the former divisor had been any other number than 6, as 5, 8, or 10, the second remainder 3 would have represented 3 parcels of 5, 8, or 10 grs. each: hence the true remainder is found by multiplying the second remainder by the first divisor, and adding in the second remainder; of course, if the former remainder be wanting, the second remainder multiplied by the first divisor will be the whole remainder: and if the second remainder be wanting, the former remainder is the true one.

Again, working fractionally, we find that the first division gives as a fractional remainder $\frac{5}{24}$ dwt.; because the required quotient is to be in dwts., and the remaining 5 grs. are $\frac{5}{24}$ of 1 dwt. The second remainder is $\frac{3}{4}$ dwt., because the 161 consists of dwts.; hence the whole remainder is $\frac{5}{24}$ dwts. + $\frac{3}{4}$ dwts. = $\frac{5+18}{24}$ dwts. = $\frac{23}{24}$ dwts.; and it will be observed *that the second* numerator 3, which is the second remainder in the former

method, was multiplied by 6 before it could be added to 5, the former numerator, *i. e.* the former remainder: hence by both processes the second remainder 3 has been multiplied by 6, so that the two methods of obtaining the remainder coincide in their operation as well as in their result.

199. We may also explain the correctness of the usual method of dividing by 20, 30, &c. or any number formed by the multiplication of 10 and any number not exceeding twelve.

Ex. To divide 3275 by 20.

Performing the division by the factors 10 and 2, as in the last article, we observe that the division by 10 gives the same result as merely cutting off the last figure as a remainder, and considering all the other numbers as the quotient; and that if there be a remainder after the second division, a figure 1 of such remainder represents not 1 but 10: hence the whole remainder is $10 + 5 = 15$.

$$20 \left\{ \begin{array}{l} 10)\ 3275 \\ 2)\ \underline{327}\ 5 \\ \quad 163\ 1 \end{array} \right\} 15$$

And the work may be written thus $\begin{array}{l} 20)\ 3275 \\ \underline{163}\ ,,\ 15, \end{array}$ or $163\frac{15}{20} = 163\frac{3}{4}$.

CIRCULATING DECIMALS.

200. Sometimes a decimal of very long period may be easily carried out to many places, without performing the division throughout, as in the following Ex.

To reduce $\frac{1}{19}$ to a decimal.

$$\begin{array}{l|l} 19)\ 1{\cdot}00\ (\cdot05263 & \text{By division we have} \\ \quad 95 & \\ \quad \overline{50} & \frac{1}{19} = {\cdot}05263\frac{3}{19} \quad \text{(H)} \\ \quad 38 & \text{therefore, multiplying both sides by 3,} \\ \quad \overline{120} & \\ \quad 114 & \frac{3}{19} = {\cdot}15789\frac{9}{19} \\ \quad \overline{60} & \\ \quad 57 & \text{and substituting for } \frac{3}{19} \text{ in (H), we have} \\ \quad \overline{3} & \\ \quad = & \frac{1}{19} = {\cdot}0526315789\frac{9}{19}; \end{array}$$

therefore $\frac{9}{19} = {\cdot}4736842101\frac{81}{19} = {\cdot}4736842105\frac{5}{19}$;

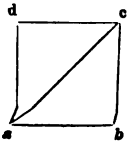
and hence $\frac{1}{19} = {\cdot}052631578947368421055\frac{5}{19}$; and by continuing this

process, it is plain that we double at every step the number of figures previously obtained. This decimal, it will be seen, circulates after the eighteenth figure; so that $\frac{1}{19} = \cdot\dot{0}5263157894736842\dot{1}$.

RATIO.

201. It was observed in (173) that quantities which cannot be accurately represented by numbers, are said to be incommensurable. And since the Exs. in Ratio and Proportion, which we have considered, have always involved only commensurable quantities, it might be thought that the application of the principles of Ratio and Proportion was limited to such quantities.

But by observing (65), from whence our definition of Ratio was taken, we learn that a ratio can exist between any two quantities whose magnitude can be represented, as A and B are, *i. e.* geometrically: and since it can be shown* that magnitudes which cannot be represented accurately in numbers, can yet be correctly represented geometrically; therefore a ratio can exist between two or more quantities, even if one or all of them be incommensurable, though the value of that ratio cannot be accurately represented in numbers. Thus, in Fig. 1, page 134, the ratio of the diagonal of a square inch to its side is represented by the fraction $\frac{a\text{——}c}{a\text{——}b}$; or arithmetically, $\frac{1{\cdot}4142....}{1}$ or $1{\cdot}4142.....$



* Some knowledge of Geometry is necessary to enable us to find the length of ac.

Since abc is a *right angle*, or one angle of a *rectangular* figure, we learn from Euclid, I. 47, that the square of which ac is the side = the sum of the squares of which ab and bc are the sides. Now, in Fig. 1, p. 134, we took $ab = 1$, and $ac = 1$; therefore the sum of the squares of ab and $ac = 1^2 + 1^2 = 2$; therefore the square of $ac = 2$, or ac itself $= \sqrt{2}$.

PROPORTION.

202. The pupil who is acquainted with Book VI. of Euclid's Elements of Geometry, will know that if two

FIG. 9.

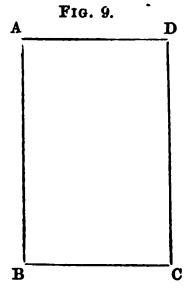
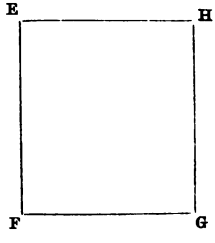


FIG. 10.



rectangular figures, AC, EG, be equal, the four sides which contain a pair of the equal angles, ABC, EFG, are proportional in the following order,—AB, EF, FG, BC; so that

$$AB : EF :: FG : BC \qquad \text{(J)}.$$

We can show that this proportion is such as would be obtained from the statement of a question in Rule of Three, in which two surfaces AC, EG, are required to be equal.

Ex. How wide a piece of cloth 15 feet long will cover a floor 13 feet 6 inches long and 10 feet wide?

Let AB be the length = 15 ft., BC the breadth, which is yet to be found; EF = 13 ft. 6 in.; FG = 10 ft.; then by the usual Statement, and writing BC as the fourth term, we have

$$15 \text{ ft.} : 13 \text{ ft. } 6 \text{ in.} :: 10 \text{ ft.} : BC;$$

and this will be found to be the same as (J).

In the language of Geometry, we say that the sides about the equal angles ABC, EFG, are *reciprocally* proportional.

203. In (138) the reader was referred to the Appendix for the solution of such a question as the following.

The shadow of a steeple is 105 yards long, and that of a stick $4\frac{1}{2}$ feet long is 15 yards: find the height of the steeple.

FIG. 11.

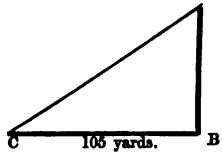
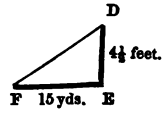


FIG. 12.



Let *AB*, *BC*, represent the steeple and its shadow; and let *DE*, *EF*, represent the stick and its shadow: I have to find the length of *AB*.

Now, most pupils, seeing this question under the head of Rule of Three, would immediately take it for granted that the three terms given would form a statement; *i. e.* that the ratio between the lengths of the two shadows is equal to that of the steeple and stick by which those shadows are cast. This is quite true; and my object in explaining this sum is merely to show what authority we have for believing that these two ratios are equal. Join *AC*, *DF*,—these lines *AC*, *DF*, will represent the direction of rays of light from the sun, and being from the same distant body are considered parallel: hence, since *AB*, *DE* are parallel, as are also *FE*, *CB*; the triangles, *ABC*, *DEF*, are said to be similar: and from a geometrical property of such triangles, we have the following:

$$EF : BC :: DE : AB;$$

or, substituting the value of *EF*, *BC*, *DE*, which are given by the question, we have

$$15 \text{ yards} : 105 \text{ yards} :: 4\tfrac{1}{2} \text{ feet} : AB;$$

and therefore $AB = \dfrac{105 \text{ yds.} \times 4\frac{1}{2} \text{ ft.}}{15 \text{ yards}} = 31\frac{1}{2} \text{ feet.}$

The following Ex. will show that it is not always safe to assume that a question which is apparently a Rule of Three Ex. will at once furnish a statement; *i. e.* that two quantities which appear to be connected, as in ordinary Exs., are really proportional to one another, as in (75) and (76).

Ex. If a ball falling from rest drop through a space of $64\frac{1}{2}$ feet in two seconds, through what distance will it have fallen in three seconds?

Now, if it were true that the distance fallen in any time were *directly proportional* to the time, the statement would be

$$2 \text{ seconds} : 3 \text{ seconds} :: 64\tfrac{1}{2} \text{ feet},$$

and the fourth term would be $96\frac{3}{4}$ feet. But a knowledge of the laws of mechanics corrects this supposition, and teaches us that the distance fallen in any time is not directly proportional to the number of seconds, but to the *square* of that number; and that in the above statement I ought to have put 2^2 and 3^2, or 4 and 9, instead of 2 and 3: hence the statement is

$$4 : 9 :: 64\tfrac{1}{2} \text{ feet};$$

and the distance fallen in three seconds $= 144\frac{9}{10}$ feet.

THE END.

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ERRATA.

Page 21, line 14, for $\frac{2}{7}$ read $\frac{2}{5}$.

Page 24, line 16, for "smaller fraction of the denominator," read "denominator of the smaller fraction."

Page 24, line 5 from bottom, for $\frac{3}{5}$ read $\frac{2}{5}$.

Page 32, line 6, take away the first 2.

Page 46, line 4, for "second by," read "secondly."

„ line 6, for "second," read "secondly."

Page 75, line 4, for (90) read (96).

Page 111, for "six," read "twelve."



